Why is $p \Rightarrow q$ defined like that?

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If you are reading this, you have already seen the following truth table definition of logical implication $p \Rightarrow q$:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

You may wonder why is it always true that F implies anything. In standard textbooks, people tend to use explanations, or rather, excuses, such as "if the promise is not broken, then it is true" or "here are examples why this makes sense".

Of course, you did NOT find them satisfactory. So some other writers say that this is for the consistency of propositional logic. But still, why?

Now, three things/"axioms" I believe that are way more convincing:

(1) *We need* $(T \Rightarrow F) = F$*.* Otherwise, mathematics becomes meaningless: you could derive false statements using true statements! So we take the second row of the above truth table as granted.

(2) *We need* Modus Ponens *(M.P.).* That is, $((p \Rightarrow q) \wedge p) \Rightarrow q$ should always be T (tautology). So whenever you have the antecedent p AND the implication $p \Rightarrow q$, you definitely have the consequent q.

(3) *We also need* $(T \Rightarrow T) = T$.

Combining (1) and (3), we get

p	q	$p \Rightarrow q$
T	T	T by (3)
T	F	F by (1)
F	T	X
F	F	Y

where X and Y are truth values (T or F) to be determined.

Claim: this is sufficient for "deriving" the "$TFTT$" definition! Indeed, let us make the truth table of $M.P.$, i.e. $((p \Rightarrow q) \wedge p) \Rightarrow q$:

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$((p \Rightarrow q) \wedge p) \Rightarrow q$
T	T	T	$T \wedge T = T$	$(T \Rightarrow T) = T$ by row 1
T	F	F	$F \wedge F = F$	$(F \Rightarrow F) = Y$ by row 4
F	T	X	$X \wedge F = F$	$(F \Rightarrow T) = X$ by row 3
F	F	Y	$Y \wedge F = F$	$(F \Rightarrow F) = Y$ by row 4

Can you see it now? Remember, by (2), $M.P.$ needs to be a tautology, *so the last column is all* T. Now row 2 to 4 dictate that we must have $X = Y = T$.

There is a reason why I put (3) at the bottom. Here is an exercise: replace (3) by the *Contrapositive Law*, i.e. $(p \Rightarrow q) = (\neg q \Rightarrow \neg p)$, and assume $(T \Rightarrow T) = Z$. Can you solve for Z using (1), (2) and Contrapositive Law?