Super Krawtchouk Polynomials via Lie Superalgebras

Joint work with Plamen Iliev

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Structure

- Introduction
- Set up
  1. $\mathfrak{gl}(m+1|n+1)$
  2. Module, form
- Orthogonality, main results [Classical results are corollaries]
- Fermionic system's probability theoretical viewpoint

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- Griffiths also proposed a multivariate version
- Iliev gave a Lie theoretic interpretation [Ili12]

An exmaple

Fix $N \in \mathbb{Z}_{\geq 0}$. For $m, x \in \{0, 1, \ldots, N\} =: [N]$ set

$$(1+z)^{N-x}(1-z)^{x} = \sum_{m=0}^{N} \binom{N}{m} \mathcal{P}_{\overline{0}}(x,m)\, z^{m}.$$

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($_{\overline{0}}$ means even)
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Explicitly one may calculate

$$\mathcal{P}_{\overline{0}}(x,0) = 1, \quad \mathcal{P}_{\overline{0}}(x,1) = \frac{1}{N}(-x+2N), \quad \mathcal{P}_{\overline{0}}(x,2) = \frac{1}{\binom{N}{2}} \left(2x^{2} - 2Nx + \binom{N}{2}\right)$$

An example, cont.

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Also: 3-term recurrence relation.

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Need $\mathrm{K} := (p, \tilde{p}, U)$:

1. scalar tuples $p, \tilde{p} \in \mathbb{C}^{m+1}$
2. complex $(m+1) \times (m+1)$ matrix U

satisfying

1. $p_0 = \tilde{p}_0 \neq 0$;
2. $U = (u_{i,j})_{i,j=0}^{m}$ satisfies the normalization condition $u_{0,j} = u_{i,0} = 1$ for $i, j \in [m]$;
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Related to character algebras.

Krawtchouk polynomials

1. $\{t_i : i \in [m]\}$: $m+1$ commuting indeterminates.
2. $D \in \mathbb{Z}_{\geq 0}$: a fixed degree parameter
3. $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m), \tilde{\alpha} = (\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) \in \mathbb{Z}_{\geq 0}^{m+1}$ such that $|\alpha| = |\tilde{\alpha}| = D$.

$$\prod_{i=0}^{m} \left(\sum_{j=0}^{m} u_{i,j} t_j \right)^{\tilde{\alpha}_i} = \sum_{\alpha} \frac{D!}{\alpha!} \mathcal{P}_{\overline{0}}(\alpha, \tilde{\alpha}) t^{\alpha}. \tag{1}$$

(so in $U = \begin{pmatrix} 1 & 1 \\ 1 & -p_0/p_1 \end{pmatrix}$ in the previous example)

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$\mathcal{P}_{\bar{0}}$ are in fact polynomials. In [MT04], an explicit formula for these polynomials is given, and the roles of α and $\tilde{\alpha}$ are completely symmetric.

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$$\psi_i \psi_j = -\psi_j \psi_i, \quad \text{for all } i, j \in [n], \quad (\Rightarrow \psi_j^2 = 0)$$

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$$\prod_{i=0}^{n} \left(\sum_{j=0}^{n} v_{i,j} \psi_j \right)^{\tilde{\epsilon}_i} = \sum_{\epsilon} D! \mathcal{P}_{\overline{1}}(\epsilon, \tilde{\epsilon}) \psi^{\epsilon}, \tag{2}$$

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Indeed, by the anticommutativity of ψ_j's, the left side reduces to certain determinant-type formula.
Explicitly: $A := \operatorname{diag}(\tilde{\epsilon})V\operatorname{diag}(\epsilon)$. Then

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Reversing the roles of the ϵ and $\tilde{\epsilon}$ amounts to transposing the matrix V.

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A uniform way to get super Krawtchouk polynomials: put K and Λ together.

Write

$$\mathbb{I}_{m|n} := \{0, 1, \ldots, m|m+1, \ldots, m+n+1\}.$$

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Introduce $\{z_k : k \in \mathbb{I}_{m|n}\}$ with $z_i := t_i$ (commutative variables), for $i \in [m]$ and $z_{m+1+j} := \psi_j$ (anticommutative) for $j \in [n]$. t_i and ψ_j commute.

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Set $(A|B) := \mathrm{diag}(A|B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Set $Y := (y_{i,j})_{i,j\in\mathbb{I}_{m|n}} = (U|V)$. Then

$$(p_0^{-1}P|q_0^{-1}Q)Y(\tilde{P}|\tilde{Q})Y^{\mathsf{t}} = I_{m+n+2}.$$

Generating Function

Let $\mathbb{A}_{m|n} := \mathbb{Z}_{\geq 0}^{m+1} \times \mathbb{Z}_2^{n+1}$, $\quad \mathbb{A}_{m|n}^D := \{(\alpha, \epsilon) \in \mathbb{A}_{m|n} : |(\alpha, \epsilon)| = D\}$.

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For $(\tilde{\alpha},\tilde{\epsilon}) \in \mathbb{A}_{m|n}^D$ we write

$$\prod_{i\in\mathbb{I}_{m|n}} \left(\sum_{j\in\mathbb{I}_{m|n}} y_{i,j} z_j \right)^{(\tilde{\alpha},\tilde{\epsilon})_i} = \sum_{(\alpha,\epsilon)\in\mathbb{A}_{m|n}^D} \frac{D!}{\alpha!} \mathcal{P}(\alpha,\epsilon,\tilde{\alpha},\tilde{\epsilon};\mathrm{K},\Lambda,D,d) z^{(\alpha,\epsilon)}. \tag{4}$$

Thus

$$\mathcal{P}(\alpha,\epsilon,\tilde{\alpha},\tilde{\epsilon}) = \binom{D}{d}^{-1} \mathcal{P}_{\overline{0}}(\alpha,\tilde{\alpha};\mathrm{K},D-d)\mathcal{P}_{\overline{1}}(\epsilon,\tilde{\epsilon};\Lambda,d). \tag{5}$$

Note $\mathcal{P}(\alpha,\epsilon,\tilde{\alpha},\tilde{\epsilon}) = 0$ unless $(\alpha,\epsilon),(\tilde{\alpha},\tilde{\epsilon}) \in \mathbb{A}_{m|n}^{D-d,d}$.

Lie superalgebra

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$$\begin{array}{c} \\ m+1 \\ n+1 \end{array} \quad \begin{array}{cc} m+1 & n+1 \\ \multicolumn{2}{c}{\left(\begin{array}{c|c} \bar{0} & \bar{1} \\ \hline \bar{1} & \bar{0} \end{array}\right)} \end{array}$$

Supercommutator:

$$[X, Y] = XY - (-1)^{|X||Y|} YX.$$

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Cartan $\mathfrak{h}$: standard diagonal; $\tilde{\mathfrak{h}}$: non-diagonal, obtained via certain change of basis using $U, V, P, Q, \tilde{P}, \tilde{Q}$ from K, Λ.
An antiautomorphism φ: also defined using K, Λ, and the supertranspose.
φ fixed both $\mathfrak{h}, \tilde{\mathfrak{h}}$.

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$\mathfrak{P}^D$: space of deg D homogeneous polynomials in commuting $\{x_i : i \in [m]\}$ and anticommuting $\{\xi_j : h \in [n]\}$.

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Can take derivatives. Example. $\xi_3 \partial_{\xi_2}(x_1\xi_1\xi_2) = x_1\xi_3(0 - \xi_1) = x_1\xi_1\xi_3$ (super Leibniz rule). Write

$$w_i := x_i, \text{ for } i \in [m]; \quad w_{m+1+j} := \xi_j \text{ for } j \in [n]; \quad \partial_i = \partial_{w_i} \text{ for } i \in \mathbb{I}_{m|n}.$$

Then $E_{i,j} \in \mathfrak{g}$ acts by

$$\rho : E_{i,j} \mapsto w_i \partial_j$$

Weight bases

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Theorem A

The super Krawtchouk polynomials $\mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) = \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}; \mathrm{K}, \Lambda, D)$ arise, up to explicit scalars, as entries of the transition matrices between the $\mathfrak{h}$- and $\tilde{\mathfrak{h}}$-weight bases for the module $\mathfrak{P}^D$. Specifically, we have

$$\tilde{x}^{\tilde{\alpha}} \tilde{\xi}^{\tilde{\epsilon}} = \tilde{\theta}^{D-d} \tilde{\kappa}^d D! \sum_{(\alpha,\epsilon) \in \mathbb{A}_{m|n}^{D-d,d}} \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) \frac{\tilde{p}^\alpha \tilde{q}^\epsilon}{\alpha!} x^\alpha \xi^\epsilon, \tag{6}$$

$$x^\alpha \xi^\epsilon = \theta^{D-d} \kappa^d D! \sum_{(\tilde{\alpha},\tilde{\epsilon}) \in \mathbb{A}_{m|n}^{D-d,d}} \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) \frac{p^{\tilde{\alpha}} q^{\tilde{\epsilon}}}{\tilde{\alpha}!} \tilde{x}^{\tilde{\alpha}} \tilde{\xi}^{\tilde{\epsilon}}. \tag{7}$$

Orthogonality

Theorem B

Let $(\alpha,\epsilon),(\tilde{\alpha},\tilde{\epsilon}),(\beta,\eta),(\tilde{\beta},\tilde{\eta}) \in \mathbb{A}_{m|n}^{D-d,d}$. *The super Krawtchouk polynomials satisfy the following orthogonality condition*

$$\sum_{(\alpha,\epsilon)\in\mathbb{A}_{m|n}^{D-d,d}} \tilde{p}^{\alpha}\tilde{q}^{\epsilon}\frac{D!}{\alpha!}\mathcal{P}(\alpha,\epsilon,\tilde{\alpha},\tilde{\epsilon})\mathcal{P}(\alpha,\epsilon,\tilde{\beta},\tilde{\eta}) = \delta_{\tilde{\alpha},\tilde{\beta}}\delta_{\tilde{\epsilon},\tilde{\eta}}\frac{\tilde{\alpha}!}{D!}\frac{p_0^{D-d}q_0^{d}}{p^{\tilde{\alpha}}q^{\tilde{\epsilon}}},$$

$$\sum_{(\tilde{\alpha},\tilde{\epsilon})\in\mathbb{A}_{m|n}^{D-d,d}} p^{\tilde{\alpha}}q^{\tilde{\epsilon}}\frac{D!}{\tilde{\alpha}!}\mathcal{P}(\alpha,\epsilon,\tilde{\alpha},\tilde{\epsilon})\mathcal{P}(\beta,\eta,\tilde{\alpha},\tilde{\epsilon}) = \delta_{\alpha,\beta}\delta_{\epsilon,\eta}\frac{\alpha!}{D!}\frac{p_0^{D-d}q_0^{d}}{\tilde{p}^{\alpha}\tilde{q}^{\epsilon}}.$$

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- On $\mathfrak{P}^D$, we set

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  WARNING: not *super*symmetric. That would mean $\langle \xi, \xi \rangle = 0$. Not what we want.
- A key point: this form is still diagonal with respect to the tilde basis $\{\tilde{x}^\beta \tilde{\xi}^\eta : (\alpha,\epsilon) \in \mathbb{A}^D_{m|n}\}$

$$\langle \tilde{x}^\alpha \tilde{\xi}^\epsilon, \tilde{x}^\beta \tilde{\xi}^\eta \rangle = \delta_{\alpha,\beta}\delta_{\epsilon,\eta}\frac{\alpha!}{p^\alpha q^\epsilon}\tilde{\theta}^{|\alpha|}\tilde{\kappa}^{|\epsilon|}, \text{ for all } (\alpha,\epsilon),(\beta,\eta) \in \mathbb{A}^D_{m|n}.$$

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Even: essentially an induction on the degree $|\alpha|$.

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$$\langle \tilde{x}^{\alpha}\tilde{\xi}^{\epsilon}, \tilde{x}^{\beta}\tilde{\xi}^{\eta} \rangle = \delta_{\alpha,\beta}\delta_{\epsilon,\eta}\frac{\alpha!}{p^{\alpha}q^{\epsilon}}\tilde{\theta}^{|\alpha|}\tilde{\kappa}^{|\epsilon|}.$$

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Definition (Multi-color super grading)

On $\mathbb{I}_{m|n} = \{0, \ldots, m|m+1, \ldots, m+n+1\}$ set $|i| = 0$ if $i \leq m$ and $|i| = 1$ if $i \geq m+1$. Let v_k denote the k-th standard basis of $\mathbb{Z}_2^{n+1}$. Whenever $k < 0$, the symbol v_k is regarded as 0. For $E_{i,j} \in \mathfrak{g}$, we set

$$\overline{E_{i,j}} := |i| v_{i-(m+1)} + |j| v_{j-(m+1)} \in \mathbb{Z}_2^{n+1}.$$

Orthogonality

Explicitly, for $i, j \in \mathbb{I}_{m|n}$

$$\overline{E_{i,j}} = \begin{cases} 0, & |i| = |j| = 0 \\ v_{i-(m+1)} + v_{j-(m+1)}, & |i| = |j| = 1 \\ v_{i-(m+1)}, & |i| = 1, |j| = 0 \\ v_{j-(m+1)} & |i| = 0, |j| = 1 \end{cases}.$$

Let $\mathfrak{g}_\epsilon := \mathrm{Span}\{E_{i,j} : \overline{E_{i,j}} = \epsilon\}$ for $\epsilon \in \mathbb{Z}_2^{n+1}$. Thus

$$\mathfrak{g} = \bigoplus_{\epsilon \in \mathbb{Z}_2^{n+1}} \mathfrak{g}_\epsilon.$$

From a physics point of view, it records which fermionic positions in $E_{i,j}$ modulo 2.

Orthogonality

The usual $\mathbb{Z}_2$-super grading is consistent with this multi-color $\mathbb{Z}_2^{n+1}$-grading:

$$[\mathfrak{g}_\epsilon, \mathfrak{g}_\eta] \subseteq \mathfrak{g}_{\epsilon+\eta}.$$

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In the sense of [Sch79], this gives a $\mathbb{Z}_2^{n+1}$-graded π-Lie algebra structure where $\pi : \mathbb{Z}_2^{n+1} \to \mathbb{C}^\times : \eta \mapsto (-1)^{|\eta|}$ is a character; compare [Pri97].

Orthogonality

The $\mathfrak{g}$-mod $\mathfrak{P}^D$ has a compatible grading too.

For $x^\alpha \xi^\epsilon \in \mathfrak{P}^D$:

$$\overline{x^\alpha \xi^\epsilon} := (\epsilon_0, \ldots, \epsilon_n) \in \mathbb{Z}_2^{n+1}, \quad \text{then} \quad \overline{E_{i,j}.x^\alpha \xi^\epsilon} = \overline{E_{i,j}} + \overline{x^\alpha \xi^\epsilon}.$$

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Proposition

For all homogeneous $X \in \mathfrak{g}$ and $u, v \in \mathfrak{P}^D$, we have

$$\langle X.u,\, v\rangle = (-1)^{|X|\overline{X}\cdot\overline{u}} \langle u, \varphi(X).v\rangle. \tag{8}$$

Here φ is a map that “captures” K, Λ and the supertranspose in $\mathfrak{g}$.

Orthogonality

Using this proposition, one can prove

$$\langle \tilde{x}^{\alpha}\tilde{\xi}^{\epsilon}, \tilde{x}^{\beta}\tilde{\xi}^{\eta}\rangle = \delta_{\alpha,\beta}\delta_{\epsilon,\eta}\text{scalar}$$

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To show Theorem B, we use Theorem A

$$\langle \tilde{x}^{\tilde{\alpha}}\tilde{\xi}^{\tilde{\epsilon}}, \tilde{x}^{\tilde{\beta}}\tilde{\xi}^{\tilde{\eta}}\rangle = \text{const.} \sum \text{scalar}\, \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon})\langle x^{\alpha}\xi^{\epsilon}, \tilde{x}^{\tilde{\beta}}\tilde{\xi}^{\tilde{\eta}}\rangle$$

The pairing $\langle x^{\alpha}\xi^{\epsilon}, \tilde{x}^{\tilde{\beta}}\tilde{\xi}^{\tilde{\eta}}\rangle = \text{scalar}\, \mathcal{P}(\alpha, \epsilon, \tilde{\beta}, \tilde{\eta})$ again by Theorem A.

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$$\text{scalar}\, \delta = \text{const.} \sum \text{scalar}\, \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon})\mathcal{P}(\alpha, \epsilon, \tilde{\beta}, \tilde{\eta}) \quad \square$$

Difference operators

Proposition 3.1

Let the notation be as above. Let $\epsilon, \tilde{\epsilon} \in \mathbb{Z}_2^{n+1}$ *with* $|\epsilon| = |\tilde{\epsilon}| = d$*. The odd Krawtchouk polynomial* $\mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon})$ *satisfies the following recursion relation*

$$\epsilon_0 \mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon}) = \sum_{k=0}^{n} q_k \tilde{\epsilon}_k \mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon}) + \sum_{0 \leq k \neq l \leq n} q_k (-1)^{s_k(\tilde{\epsilon}-v_l)+s_l(\tilde{\epsilon})} \tilde{\epsilon}_l \mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon} + v_k - v_l), \tag{9}$$

$$\tilde{\epsilon}_0 \mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon}) = \sum_{k=0}^{n} \tilde{q}_k \epsilon_k \mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon}) + \sum_{0 \leq k \neq l \leq n} \tilde{q}_k (-1)^{s_k(\epsilon-v_l)+s_l(\epsilon)} \epsilon_l \mathcal{P}_{\bar{1}}(\epsilon + v_k - v_l, \tilde{\epsilon}). \tag{10}$$

$$\epsilon_i \mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon}) = q_0^{-1} \tilde{q}_i \sum_{0 \leq k,l \leq n} q_k v_{k,i} v_{l,i} (-1)^{s_k(\tilde{\epsilon}-v_l)+s_l(\tilde{\epsilon})} \tilde{\epsilon}_l \mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon} + v_k - v_l).$$

Like recurrence relations.

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Goal: odd Krawtchouk $\rightsquigarrow$ Fermionic prob (story 1) & zonal spherical function (story 2)

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$$\mathcal{F} = \bigoplus_{j=0}^{n+1} \bigwedge^{j} \mathcal{W}.$$

$\mathcal{F}^{(d)} = \bigwedge^d \mathcal{W}$: state space of an ensemble of d identical fermions.

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Antisymmetry of the exterior products = Pauli exclusion principle ($e_i \wedge e_i = 0$)

Oriented Grassmannian

- For any oriented orthonormal basis $\{f_0, f_1, \ldots, f_{d-1}\}$ for $\mathcal{W}$, $f_0 \wedge f_1 \wedge \cdots \wedge f_{d-1}$ uniquely corresponds to an oriented d-dimensional subspace of $\mathcal{W}$ spanned by f_i's.

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- Let $G := SO(n+1)$ and $K := SO(d) \times SO(n+1-d)$.

$$\widetilde{Gr}(n+1, d) := SO(n+1)/SO(d) \times SO(n+1-d) = G/K.$$

$\neq 0$ simple wedges (up to a positive scalar) naturally corresponds to points in $\widetilde{Gr}(n+1, d)$.

Story 1: Fermion Probabilities

• For $I = \{i_k : i_0 < i_1 < \cdots < i_{d-1}\} \subseteq [n]$ set $e_I := e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge e_{i_{d-1}}$. Then for any $g \in GL$

$$g.e_J = \sum_{|I|=d} \det(g_{I,J}) e_I. \tag{11}$$

(coefficients are the Plücker coordinates of $g.P_J = \mathrm{Span}\{e_j : j \in J\}$)
• Fix $|J| = d$. For $g \in SO(n+1)$ the vector $g.e_J$ as in Eq. (11) has norm 1. Hence

$$\mathbb{P}_{I|J} := |\det(g_{I,J})|^2 \tag{12}$$

form a probability distribution on the set of size-d index subsets $\{I \subseteq [n] : |I| = d\}$.
• In the fermionic interpretation of $\wedge^d \mathcal{W}$, this is the probability distribution obtained by expressing the state $g.e_J$ in the occupation basis $\{e_I : |I| = d\}$, cf. [Got07, Section 2].

Story 2: Rep theory

• $\mathcal{W} = \mathbb{R}^{n+1}$: natural $G = SO(n+1)$-mod. G acts on $\mathcal{F}$ via

$$g.(u_0 \wedge \cdots \wedge u_{d-1}) := (g.u_0) \wedge \cdots \wedge (g.u_{d-1}).$$

• Our pair $(G, K) = (SO(n+1), SO(d) \times SO(n+1-d))$: G-mod $\bigwedge^d \mathcal{W}$ is typically irreducible, except a middle case when $n+1$ is even and $d = \dfrac{n+1}{2}$. Here $\bigwedge^d \mathcal{W}$ splits into two irreducible components.

• Consider the vector

$$v^K := e_0 \wedge e_1 \wedge \cdots \wedge e_{d-1} \in \bigwedge^d \mathcal{W}.$$

It is K-fixed, since for $(k_0, k_1) \in SO(d) \times SO(n+1-d)$,

$$(k_0, k_1).v^K = k_0(e_0 \wedge \cdots \wedge e_{d-1}) = (\det k_0)v^K = v^K.$$

Up to const unique. When $d = \frac{n+1}{2} \in \mathbb{Z}_{>0}$, there are two irred. components. Another K-fixed vector $v_-^K := e_d \wedge e_{d+1} \wedge \cdots \wedge e_n$; we choose the component containing v^K.

Zonal spherical function

- The associated zonal spherical function is

$$\phi_d(g) := \langle v^K,\ g.v^K \rangle, \qquad g \in G,$$

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- G acts transitively on oriented bases, may choose $\sigma_I, \sigma_J \in G$ such that

$$\sigma_I.v^K = e_I, \quad \sigma_J.v^K = e_J.$$

It is not hard to see that the (I, J)-minor of g is exactly given by $\langle e_I, g.e_J \rangle$.

- In terms of ϕ_d:

$$\begin{aligned}\langle e_I, g.e_J\rangle &= \langle \sigma_I v^K, g\sigma_J.v^K\rangle \\ &= \langle v^K, \sigma_I^{-1} g\sigma_J.v^K\rangle \\ &= \phi_d(\sigma_I^{-1} g\sigma_J).\end{aligned} \tag{14}$$

Choice-independent by the bi-K-invariance!

- Combing everything all together,

$$\mathcal{P}_{\overline{1}}(\epsilon, \tilde{\epsilon}) = \frac{q_0^{d/2}}{d!} \sum_{|I|=|J|=d} q_I^{-\frac{1}{2}} \tilde{q}_J^{-\frac{1}{2}} \phi_d(\sigma_I^{-1} g_V \sigma_J)\tilde{\epsilon}_I \epsilon_J. \tag{15}$$

Coefficients of $\mathcal{P}_{\overline{1}}(\epsilon, \tilde{\epsilon})$ = weighted evaluation of ϕ_d of $\widetilde{Gr}(n+1, d)$.

Interpretation

- $$\mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon}) = \frac{q_0^{d/2}}{d!} \sum_{|I|=|J|=d} q_I^{-\frac{1}{2}} \tilde{q}_J^{-\frac{1}{2}} \phi_d(\sigma_I^{-1} g_V \sigma_J) \tilde{\epsilon}_I \epsilon_J. \tag{16}$$

In view of the probability interpretation (story 1) above (Eq. (12)), the squared norms of the evaluation of ϕ_d are the transition probabilities of different occupation basis dictated by $\Lambda = (q, \tilde{q}, V)$.

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