Supersymmetric Shimura operators and interpolation polynomials

Joint work with Siddhartha Sahi

Songhao Zhu

Rutgers University
sz446@math.rutgers.edu

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Structure

- Background (S. Sahi and G. Zhang [SZ19])
  - Shimura operators
  - Okounkov Polynomials
- Main Results and Ideas
  - Super ingredients (supersymmetric Shimura operators, Sergeev–Veselov polynomials)
  - Three Theorems
- Future Directions

Background

1. $X = G/K$: rank n symmetric space. $\mathfrak{D} = \mathfrak{D}(X)$: space of invariant differential operators on X.
2. $(\mathfrak{g}, \mathfrak{k})$: corresponding Lie algebras. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$: Iwasawa decomposition.
3. $\gamma^0 : \mathfrak{D}(X) \to \Lambda \subseteq \mathfrak{P}(\mathfrak{a}^*)$: the Harish-Chandra isomorphism. Λ is a certain ring of symmetric polynomials.
4. Shimura [Shi90]: multivariate generalization of nearly holomorphic forms. Studied certain differential operators on *Hermitian* X.

$$\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+ (= \mathfrak{k} \oplus \mathfrak{p})$$

   Short grading, $\mathfrak{k}$ acts on $\mathfrak{p}^\pm$ (abelian).
5. Sahi–Zhang described the spectrum of these *Shimura operators* in terms of specialization of BC-symmetric interpolation polynomials by Okounkov. [Realized as the images under γ^0.]

Schmid Decomposition and Shimura Operators

Let $\mathscr{H}(n)$ consist of partitions of length n, $\mathscr{H}^d(n) := \{\lambda \in \mathscr{H}(n) : |\lambda| = d\}$. Denote $\mathfrak{U}(\mathfrak{g})$ as $\mathfrak{U}$, and the $\mathfrak{k}$ centralizer in $\mathfrak{U}$ as $\mathfrak{U}^{\mathfrak{k}}$. Then $\mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$ where $(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}} = \mathfrak{U}\mathfrak{k} \cap \mathfrak{U}^{\mathfrak{k}}$. A result of Schmid ([Sch70, FK90]) gives the following multiplicity free $\mathfrak{k}$-module decompositions:

$$\mathfrak{S}^d(\mathfrak{p}^+) = \bigoplus_{\lambda \in \mathscr{H}^d(n)} W_\lambda, \ \mathfrak{S}^d(\mathfrak{p}^-) = \bigoplus_{\lambda \in \mathscr{H}^d(n)} W_\lambda^*.$$

Shimura Operators

$$\begin{array}{c} \mathrm{End}_{\mathfrak{k}}(W_\lambda) \cong (W_\lambda^* \otimes W_\lambda)^{\mathfrak{k}} \hookrightarrow \left(\mathfrak{S}(\mathfrak{p}^-) \otimes \mathfrak{S}(\mathfrak{p}^+)\right)^{\mathfrak{k}} \to \mathfrak{U}^{\mathfrak{k}} \to \mathfrak{D} \\ 1 \longmapsto D_\lambda \mapsto \mathscr{D}_\lambda \end{array}$$

Call $\mathscr{D}_\lambda$ the *Shimura operator associated with* λ.

Okounkov Polynomials

$\Lambda := \mathbb{C}[x_1, \ldots, x_n]^{S_n \ltimes \mathbb{Z}_2^n}$ (ring of even symmetric polynomials)
$\rho := (\rho_1, \ldots, \rho_n)$, $\rho_i := \tau(n-i) + \alpha$. τ, α: parameters.

Theorem-Definition [Oko98, OO06], c.f. [SZ19]

The Okounkov polynomial $P_\mu(x; \tau, \alpha)$ is the unique polynomial in Λ satisfying

1. $\deg P_\mu = 2|\mu|$;
2. $P_\mu(\lambda + \rho) = 0$ for $\lambda \not\supseteq \mu$ [**the vanishing properties**];
3. Some normalization condition.

ρ can be specialized to the half sum of positive roots for a restricted root system of Type BC. [The case for Hermitian X]
For the "usual" Type A symmetry, there are Knop–Sahi polynomials [KS96].

Punch Line

Let W_0 be the Weyl group of the restricted root system. For Hermitian G/K, $W_0 \cong S_n \ltimes \mathbb{Z}_2^n$. Then $\Lambda = \mathfrak{P}(\mathfrak{a}^*)^{W_0}$ and $\gamma^0 : \mathfrak{D} \to \mathfrak{P}(\mathfrak{a}^*)^{W_0}$.

$$\begin{aligned} \mathrm{End}_{\mathfrak{k}}(W_\lambda) \to \mathfrak{D} &\xrightarrow{\gamma^0} \mathfrak{P}(\mathfrak{a}^*)^{W_0} \\ 1 &\longmapsto \gamma^0(\mathscr{D}_\lambda) \end{aligned}$$

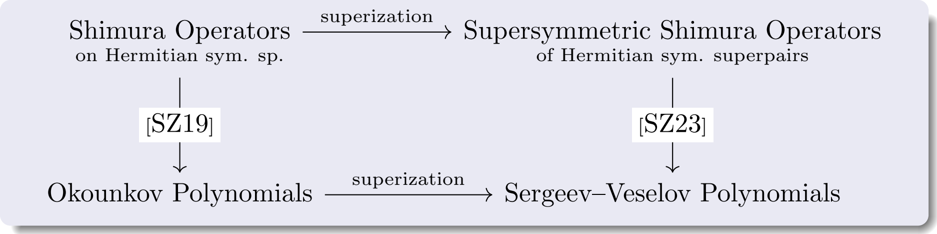
Theorem (Sahi & Zhang [SZ19])

We have $\gamma^0(\mathscr{D}_\lambda) = k_\lambda P_\lambda$ for some $k_\lambda \neq 0$.

Let V_μ be the irreducible $\mathfrak{g}$-module of highest weight $\sum \mu_i \gamma_i$. Then V_μ has a spherical vector $v^{\mathfrak{k}}$, i.e. $\mathfrak{k}.v^{\mathfrak{k}} = 0$. This is guaranteed by the classic Cartan–Helgason Theorem. $D_\lambda \in \mathfrak{U}^{\mathfrak{k}}$ ($\mathscr{D}_\lambda \in \mathfrak{D}$) acts on $v^{\mathfrak{k}}$ as $\gamma^0(\mathscr{D}_\lambda)(\mu + \rho)$, hence the word spectrum/eigenvalue!

Big Picture

We solved the Type A super analog:



Set up

Fix $\mathfrak{g} = \mathfrak{gl}(2p|2q)$ and $\mathfrak{k} = \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q)$. Embed $\mathfrak{k}$ into $\mathfrak{g}$:

$$\left(\left(\begin{array}{c|c} A_{p\times p} & B_{p\times q} \\ \hline C_{q\times p} & D_{q\times q} \end{array} \right), \left(\begin{array}{c|c} A'_{p\times p} & B'_{p\times q} \\ \hline C'_{q\times p} & D'_{q\times q} \end{array} \right) \right) \mapsto \left(\begin{array}{cc|cc} A_{p\times p} & 0_{p\times p} & B_{p\times q} & 0_{p\times q} \\ 0_{p\times p} & A'_{p\times p} & 0_{p\times q} & B'_{p\times q} \\ \hline C_{q\times p} & 0_{q\times p} & D_{q\times q} & 0_{q\times q} \\ 0_{q\times p} & C'_{q\times p} & 0_{q\times q} & D'_{q\times q} \end{array} \right)$$

Here $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) consists of matrices with non-zero entries only in the upper right (resp. bottom left) sub-blocks in each of the four blocks.
Let $J := \frac{1}{2}\operatorname{diag}(I_{p\times p}, -I_{p\times p}, I_{q\times q}, -I_{q\times q})$, and $\theta := \operatorname{Ad}\exp(i\pi J)$. Then θ has fixed point subalgebra $\mathfrak{k}$.
Fix a θ-stable, maximally split Cartan $\mathfrak{h}$ containing $\mathfrak{a}$, a maximal toral subalgebra in $\mathfrak{p}_{\bar{0}}$. The standard diagonal Cartan is denoted as $\mathfrak{t}$ (in both $\mathfrak{g}$ and $\mathfrak{k}$, "max. compact")

Super Ingredients

Let $\mathscr{H} = \mathscr{H}(p,q) := \{\lambda : \lambda_{p+1} \leq q\}$ (hook partitions), and let $\mathscr{H}^d := \{\lambda \in \mathscr{H} : |\lambda| = d\}$. For $\mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q)$, Cheng–Wang decomposition ([CW01, SSS20]) says

$$\mathfrak{S}^d(\mathfrak{p}^+) = \bigoplus_{\lambda \in \mathscr{H}^d} W_\lambda, \; \mathfrak{S}^d(\mathfrak{p}^-) = \bigoplus_{\lambda \in \mathscr{H}^d} W_\lambda^*.$$

Highest weight ($\lambda^\natural$) on $\mathfrak{t}$, expressed in Harish-Chandra strongly orthogonal roots. Note W_λ are of Type $\mathtt{M}$, and $\dim \operatorname{End}_{\mathfrak{k}}(W_\lambda) = 1$. Set $\mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$.

Supersymmetric Shimura Operators

$$\begin{array}{ccccccccc} \operatorname{End}_{\mathfrak{k}}(W_\lambda) \cong (W_\lambda^* \otimes W_\lambda)^{\mathfrak{k}} \hookrightarrow \left(\mathfrak{S}(\mathfrak{p}^-) \otimes \mathfrak{S}(\mathfrak{p}^+)\right)^{\mathfrak{k}} \to \mathfrak{U}^{\mathfrak{k}} \to \mathfrak{D} \\ 1 \longmapsto D_\lambda \mapsto \mathscr{D}_\lambda \end{array}$$

Call $\mathscr{D}_\lambda$ the *supersymmetric Shimura operator associated with* λ.

Super Ingredients

Iwasawa decomposition (for recent developments see [She22]) $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ gives

$$\mathfrak{U} = (\mathfrak{U}\mathfrak{k} + \mathfrak{n}\mathfrak{U}) \oplus \mathfrak{S}(\mathfrak{a}).$$

Then the homomorphism Γ (Harish-Chandra *homomorphism*) is defined as the ρ-shifted projection w.r.t. the above decomposition. The quotient *isomorphism* $\gamma^0 : \mathfrak{U}^{\mathfrak{k}} / \ker \Gamma \to \operatorname{Im} \Gamma$ is the *Harish-Chandra isomorphism*.

1. Independent from Alldridge's results on Harish-Chandra homomorphism [All12], we proved $\ker \Gamma = (\mathfrak{U}\mathfrak{k})^{\mathfrak{k}} := \mathfrak{U}\mathfrak{k} \cap \mathfrak{U}^{\mathfrak{k}}$.
2. We also proved that $\operatorname{Im} \gamma^0$ is exactly $\Lambda^0(\mathfrak{a}^*)$, the ring of even supersymmetric polynomials on $\mathfrak{a}^*$, previously proved in [Zhu22].
   Even supersymmetric: invariant under permutations of $\{x_i\}$ and of $\{y_j\}$ separately; invariant under sign changes of $\{x_i, y_j\}$; and $f(x_1 = t, y_1 = -t)$ is independent of t.

Sergeev–Veselov Polynomials

Proposition-Definition

For each $\mu \in \mathscr{H}$, there is a unique polynomial $J_\mu \in \Lambda^0$ of degree $2|\mu|$ s.t.

$$J_\mu(\overline{\lambda} + \rho) = 0, \quad \text{for all } \lambda \not\supseteq \mu,\ \lambda \in \mathscr{H}$$

and that $J_\mu(\overline{\mu} + \rho)$ is certain explicit non-zero constant.

1. A specialization of Sergeev–Veselov polynomials [SV09].
2. Here $\overline{\lambda}$ is some choice of coordinates (*Frobenius*).
3. ρ is the Weyl vector, the half sum of the positive restricted roots.

E.g. $p = q = 1$, for the restricted root system, $\rho = (-1, 1)$.

1. $\mu = (1)$, $\lambda = \varnothing$, and $\overline{\lambda} + \rho = (-1, 1)$, $J_{(1)} \propto x^2 - y^2$.
2. $\mu = (2)$, $\lambda = (1^n)$, and $\overline{\lambda} + \rho = (1, 2n - 1)$, $J_{(2)} \propto (x^2 - y^2)(x^2 - 1)$.

Main Results

Theorem A (Sahi & Z. [SZ23])

We have $\gamma^0(\mathscr{D}_\mu) = k_\mu J_\mu$ *where* $k_\mu = (-1)^{|\mu|} \prod_{(i,j)\in\mu} \left(\mu_i - j + \mu'_j - i + 1\right)$.

The main thing is to show the vanishing properties. Need two other results. Let the center of $\mathfrak{U}$ be $\mathfrak{Z}$. Then $\mathfrak{Z} \subseteq \mathfrak{U}^{\mathfrak{k}}$ and we have $\pi : \mathfrak{Z} \hookrightarrow \mathfrak{U}^{\mathfrak{k}} \twoheadrightarrow \mathfrak{D}$.

Theorem B (Sahi & Z. [SZ23])

The map π *is surjective. In particular, there exist* $Z_\mu \in \mathfrak{Z}$ *such that* $\pi(Z_\mu) = \mathscr{D}_\mu$. (So $\mathscr{D}_\mu = \pi(D_\mu)$ can be captured by some central element!)

Let $I_\lambda := \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda$ be the generalized Verma module for $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$.

Theorem C (Sahi & Z. [SZ23])

The central element Z_μ *acts on* I_λ *by* 0 *when* $\lambda \not\supseteq \mu$.

Main Results

Why the fuss?

1. No full generalization of Cartan–Helgason theorem in the super scenario. The only partial result is obtained by Alldridge and Schmittner [AS15]. Not enough V_λ are guaranteed to be spherical.
2. Only know how $\mathfrak{U}^{\mathfrak{k}}$ (or $\mathfrak{D}$) acts on a spherical vector of an irreducible, finite dimensional, $\mathfrak{h}$-highest weight $\mathfrak{g}$-module V_λ. By [Zhu22, Theorem 5.2], $D \in \mathfrak{U}^{\mathfrak{k}}$ ($\pi(D) \in \mathfrak{D}$) acts on a spherical vector $v^{\mathfrak{k}}$ by the scalar $\Gamma(D)(\lambda + \rho) = \gamma^0(\pi(D))(\lambda + \rho)$.
3. I_λ has the irreducible quotient isomorphic to V_λ! We devise a workaround using this.

Main Results

Theorem B

The map π is surjective. In particular, there exist $Z_\mu \in \mathfrak{Z}$ such that $\pi(Z_\mu) = \mathscr{D}_\mu$.

$\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t}_+$: Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$
$\gamma : \mathfrak{Z} \to \mathfrak{P}(\mathfrak{h}^*)$: the usual Harish-Chandra isomorphism
Res: the restriction map induced from the decomposition $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}_+$.

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{\pi} & \mathfrak{D} \\ \gamma \downarrow \wr & & \wr \downarrow \gamma^0 \\ \Lambda(\mathfrak{h}^*) & \xrightarrow{\texttt{Res}} & \Lambda^0(\mathfrak{a}^*) \end{array}$$

1. First show $\Lambda(\mathfrak{h}^*)$ surjects onto $\operatorname{Im}\gamma^0 = \Lambda^0(\mathfrak{a}^*)$, via Res.
2. Then show the diagram commutes.

Main Results

Sketch of proof of Theorem B.

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{\pi} & \mathfrak{D} \\ \gamma \downarrow \wr & & \wr \downarrow \gamma^0 \\ \Lambda(\mathfrak{h}^*) & \xrightarrow{\texttt{Res}} & \Lambda^0(\mathfrak{a}^*) \end{array}$$

1. Choose explicit coordinates on $\mathfrak{h}^*$ and $\mathfrak{a}^*$, and explicit generators of the algebra of supersymmetric polynomials ([Ste85]) to show the surjectivity of `Res`.
2. Diagram chase. The set of the highest $\mathfrak{h}$-weights that guarantee to give a spherical irreducible $\mathfrak{g}$-module is *Zariski dense* (by [AS15], a partial generalization of the Cartan–Helgason Theorem).

□

Main Results

Theorem C

The central element Z_μ acts on I_λ by 0 when $\lambda \not\supseteq \mu$.

Sketch of the proof of Theorem C.

1. $I_\lambda = \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda \cong \mathfrak{S}(\mathfrak{p}^-) \otimes W_\lambda \cong \bigoplus (W_\mu^* \otimes W_\lambda)$ (as $\mathfrak{k}$-modules).
2. Spherical: $I_\lambda^{\mathfrak{k}} \subseteq W_\lambda^* \otimes W_\lambda$ with $\dim I_\lambda^{\mathfrak{k}} = 1$.
3. Rep map $W_\mu \otimes I_\lambda^{\mathfrak{k}} \to I_\lambda$ has image homomorphic to W_μ.
4. $\mathrm{Hom}_{\mathfrak{k}}(W_\mu, I_\lambda) = \{0\}$ for $\lambda \not\supseteq \mu$.
5. $D_\mu = \sum \xi_i \eta_i$ for $\xi_i \in W_\mu^*$ and $\eta_i \in W_\mu$. So $D_\mu . I_\lambda^{\mathfrak{k}} = \{0\} = \mathscr{D}_\mu . I_\lambda^{\mathfrak{k}}$.
6. Z_μ also acts by 0. But $Z_\mu \in \mathfrak{Z}$ so it acts by 0 everywhere!

□

The main thing is that I_λ has $\mathfrak{t}$-highest weight and is infinite dimensional. We don't know by what "polynomial" $\mathscr{D}_\mu$ acts on $I_\lambda^{\mathfrak{k}}$ directly!

Main Results

Theorem A

We have $\gamma^0(\mathscr{D}_\mu) = k_\mu J_\mu$ where $k_\mu = (-1)^{|\mu|} \prod_{(i,j)\in\mu} \left(\mu_i - j + \mu'_j - i + 1\right)$.

Sketch of the proof of Theorem A.

1. By the commutative diagram, we have $\gamma^0(\pi(Z_\mu)) = \mathtt{Res}(\gamma(Z_\mu))$. The LHS is just $\gamma^0(\mathscr{D}_\mu)$.
2. $\gamma^0(\mathscr{D}_\mu)(\overline{\lambda} + \rho) = \gamma(Z_\mu)(\lambda + \rho)$
3. By Theorem C, Z_μ acts by 0 on I_λ for $\lambda \not\supseteq \mu$. But $\mathfrak{Z}$ acts on a cyclic module exactly by γ. Thus
$$\gamma^0(\mathscr{D}_\mu)(\overline{\lambda} + \rho) = 0, \text{for all } \lambda \not\supseteq \mu,\ \lambda \in \mathscr{H}.$$
4. We use the theory of super Jack polynomials ([SV05]) to pin down k_μ by comparing leading terms.

□

Other Types

- Supersymmetric Shimura operators can be defined for other pairs (Jordan superalgebras+TKK construction) c.f. [SSS20].
- The main difficulty is perhaps the surjectivity of `Res` map and the commutative diagram which in the current setting are proved by some particular choice of coordinates. We believe this can be done in a better way.
- We would also like to generalize the Cartan–Helgason Theorem for the super setting. Appears to be difficult... [Zhu22]

Scope of the theory

$\ddot{\smile}$: usual Lie algebras. $\mathbb{Z}_2$: Lie superalgebras. q: quantum groups.

	Shimura	Capelli	quadratic Capelli
$\ddot{\smile}$	[SZ19]	[KS93, Sah94]	[SS19]
$\mathbb{Z}_2$	[Zhu22, SZ23]	[SSS20]	?
q	?	[LSS22]	?
$\mathbb{Z}_2, q$	?	?	?

Table: Scope

Thank you!

A. Alldridge, *The Harish-Chandra isomorphism for reductive symmetric superpairs*, Transform. Groups **17** (2012), no. 4, 889–919. MR 3000475

Alexander Alldridge and Sebastian Schmittner, *Spherical representations of Lie supergroups*, J. Funct. Anal. **268** (2015), no. 6, 1403–1453. MR 3306354

Shun-Jen Cheng and Weiqiang Wang, *Howe duality for Lie superalgebras*, Compositio Math. **128** (2001), no. 1, 55–94. MR 1847665

J. Faraut and A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), no. 1, 64–89. MR 1033914

Bertram Kostant and Siddhartha Sahi, *Jordan algebras and Capelli identities*, Invent. Math. **112** (1993), no. 3, 657–664. MR 1218328

Friedrich Knop and Siddhartha Sahi, *Difference equations and symmetric polynomials defined by their zeros*, Internat. Math. Res. Notices (1996), no. 10, 473–486. MR 1399412

Gail Letzter, Siddhartha Sahi, and Hadi Salmasian, *The Capelli eigenvalue problem for quantum groups*, 2022, arXiv.

A. Okounkov, BC-*type interpolation Macdonald polynomials and binomial formula for Koornwinder polynomials*, Transform. Groups **3** (1998), no. 2, 181–207. MR 1628453

Andrei Okounkov and Grigori Olshanski, *Limits of BC-type orthogonal polynomials as the number of variables goes to infinity*, Jack, Hall-Littlewood and Macdonald polynomials, Contemp. Math., vol. 417, Amer. Math. Soc., Providence, RI, 2006, pp. 281–318. MR 2284134

Siddhartha Sahi, *The spectrum of certain invariant differential operators associated to a Hermitian symmetric space*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 569–576. MR 1327549

Wilfried Schmid, *Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen*, Invent. Math. **9** (1969/70), 61–80. MR 259164

Alexander Sherman, *Iwasawa decomposition for Lie superalgebras*, J. Lie Theory **32** (2022), no. 4, 973–996. MR 4512155

Goro Shimura, *Invariant differential operators on Hermitian symmetric spaces*, Ann. of Math. (2) **132** (1990), no. 2, 237–272. MR 1070598

Siddhartha Sahi and Hadi Salmasian, *Quadratic Capelli operators and Okounkov polynomials*, Ann. Sci. Éc. Norm. Supér. (4) **52** (2019), no. 4, 867–890. MR 4038454

Siddhartha Sahi, Hadi Salmasian, and Vera Serganova, *The Capelli eigenvalue problem for Lie superalgebras*, Math. Z. **294** (2020), no. 1-2, 359–395. MR 4054814

John R. Stembridge, *A characterization of supersymmetric polynomials*, J. Algebra **95** (1985), no. 2, 439–444. MR 801279

A. N. Sergeev and A. P. Veselov, *Generalised discriminants, deformed Calogero-Moser-Sutherland operators and super-Jack polynomials*, Adv. Math. **192** (2005), no. 2, 341–375. MR 2128703

Alexander N. Sergeev and Alexander P. Veselov, *BC_∞ Calogero-Moser operator and super Jacobi polynomials*, Adv. Math. **222** (2009), no. 5, 1687–1726. MR 2555909

Siddhartha Sahi and Genkai Zhang, *Positivity of Shimura operators*, Math. Res. Lett. **26** (2019), no. 2, 587–626. MR 3999556

Siddhartha Sahi and Songhao Zhu, *Supersymmetric Shimura operators and interpolation polynomials*, 2023, https://arxiv.org/abs/2312.08661.

Songhao Zhu, *Shimura operators for certain Hermitian symmetric superpairs*, 2022, https://arxiv.org/abs/2212.09249.