Eigenvalues of Shimura Operators for Lie Superalgebras

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Structure

- Background (S. Sahi and G. Zhang [SZ19])
- Super Ingredients
- Results in [Zhu]
- Main Ideas, and one open problem

Background

Geometric Data: An irreducible Hermitian symmetric space G/K of rank n.
Algebraic Data (can be "superized"):

- (Complexified) Lie algebra pair $(\mathfrak{g}, \mathfrak{k})$ admitting the *Harish-Chandra Decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$$

- Two Cartan subalgebras $\mathfrak{t}, \mathfrak{h}$ of the pair.
  - $\mathfrak{t}$ is the "usual" one contained in both $\mathfrak{k}$ and $\mathfrak{g}$; $\mathfrak{h}$ is extended from the Cartan subspace $\mathfrak{a}$ of $\mathfrak{p}$.
- Root system $\Sigma(\mathfrak{g}, \mathfrak{t})$, with a special subset of the roots in $\Sigma(\mathfrak{p}^+, \mathfrak{t})$, $\{\gamma_i\}$, called the *Harish-Chandra roots.*
  - Used to construct $\mathfrak{a}$ and Cayley Transforms.
- Restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. Always of Type BC! Let's denote the Weyl group of $\Sigma(\mathfrak{g}, \mathfrak{a})$ as W_0.

Schmid Decomposition and Shimura Operators

Let $\mathscr{H}(n)$ be the set of partitions of length n. By a theorem of Schimid ([Sch70, FK90]), $\mathfrak{S}(\mathfrak{p}^{\pm})$ are completely reducible and multiplicity free as $\mathfrak{k}$-modules

$$\mathfrak{S}(\mathfrak{p}^+) = \bigoplus_{\lambda \in \mathscr{H}(n)} W(\lambda),\ \mathfrak{S}(\mathfrak{p}^-) = \bigoplus_{\lambda \in \mathscr{H}(n)} W^*(\lambda)$$

Here $W(\lambda)$ has highest weight $\sum \lambda_i \gamma_i$. Note $\mathfrak{p}^{\pm}$ are abelian. So $\mathfrak{S}(\mathfrak{p}^-) \otimes \mathfrak{S}(\mathfrak{p}^+)$ multiplies into $\mathfrak{U}(\mathfrak{p}^-)\mathfrak{U}(\mathfrak{p}^+) \subseteq \mathfrak{U}(\mathfrak{g})$ by the PBW theorem. Now consider the following composition of maps:

$$\begin{array}{ccc} \mathrm{End}_{\mathfrak{k}}(W(\lambda)) \cong (W^*(\lambda) \otimes W(\lambda))^{\mathfrak{k}} \hookrightarrow \left(\mathfrak{S}\left(\mathfrak{p}^-\right) \otimes \mathfrak{S}\left(\mathfrak{p}^+\right)\right)^{\mathfrak{k}} \to \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}} \\ 1 \longmapsto D_\lambda \end{array}$$

D_λ is in $\mathfrak{U}(\mathfrak{g})^{\mathfrak{k}} = \mathfrak{U}(\mathfrak{g})^K \cong \mathbf{D}_K(G)$, the space of right K-invariant differential operators on G. But it further descends to $\mathbf{D}(G/K)$. The image is called the Shimura operator.

By a slight abuse of name, we also call D_λ the *Shimura operator associated to* λ.

Okounkov Polynomials

Let $P_n = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials in n variables. W_0 acts naturally on it by permutations and sign changes. Let $\mathcal{Q} = P_n^{W_0}$ be the subalgebra of even symmetric polynomials. We also define $\rho = (\rho_1, \ldots, \rho_n)$ with $\rho_i = \tau(n-i) + \alpha$ where τ, α are two parameters.

Theorem ([Oko98], [OO06], c.f.[SZ19])

The Okounkov polynomial $P_\mu(x; \tau, \alpha)$ is the unique polynomial in $\mathcal{Q}$ satisfying

1. $\deg P_\mu = 2|\mu|$;
2. $P_\mu(\lambda + \rho) = 0$ *for* $\lambda \not\supseteq \mu$ [**the vanishing condition**];
3. *Some normalization condition.*

We remark that ρ can be specialized to the half sum of positive restricted roots for a root system of Type BC, say, $\Sigma(\mathfrak{g}, \mathfrak{a})$. Also, for the usual Type A symmetry, there are Knop–Sahi polynomials [KS96].

The Theorem

Recall $D_\mu \in \mathfrak{U}^{\mathfrak{k}}$. Now consider the Harish-Chandra homomorphism $\Gamma : \mathfrak{U}^{\mathfrak{k}} \to \mathfrak{S}(\mathfrak{a})^{W_0} \cong \mathfrak{P}(\mathfrak{a}^*)^{W_0}$.

Theorem ([SZ19])

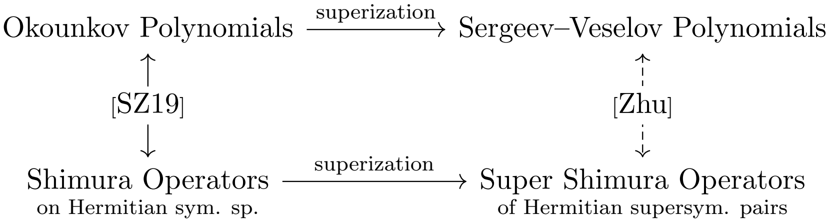
$\Gamma(D_\mu) = k_\mu P_\mu$ *for some* $k_\mu \neq 0$.

We point out that k_μ can be explicitly written down, depending only on the partition μ.

In [Zhu], I obtained a super analog of this result.

Big Picture

Specifically, we aim to show the eigenvalues of the super Shimura operators are up to constant equal to Type BC supersymmetric interpolation polynomials developed by Sergeev and Veselov [SV09]. Here is a diagram sketching the main idea:



Things to address...

- How do we superize G/K? *We superize the "algebra data"* $(\mathfrak{g}, \mathfrak{k})$.
- Is there a "super$\Gamma : \mathfrak{U}^{\mathfrak{k}} \to \mathfrak{S}(\mathfrak{a})^{W_0} \cong \mathfrak{P}(\mathfrak{a}^*)^{W_0}$"? *Yes.*$^{\textit{conditions may apply}}$
- Are Sergeev–Veselov polynomials live in $\operatorname{Im}\Gamma$? *Yes.*

Lie Superalgebras

General Principle of Superization

A (good) $\mathbb{Z}_2$-grading for everything!

$$\mathbb{Z}_2 = \{\overline{0}, \overline{1}\} = \{\texttt{even}, \texttt{odd}\}$$

Lie Superalgebras

Definition

A *vector superspace* V is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. A vector $v \in V_{\bar{0}}$ (resp. $V_{\bar{1}}$) is said to be *even* (resp. *odd*) and write $|v| = 0$ (resp. 1). Denote the vector superspace with even subspace $\mathbb{C}^m$ and odd subspace $\mathbb{C}^n$ as $\mathbb{C}^{m|n}$.

Definition ([Kac77])

A *Lie superalgebra* is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which is skew supersymmetric and satisfies the super Jacobi identity, that is

1. $[X,Y] = -(-1)^{|X||Y|}[Y,X]$
2. $[[X,Y],Z] = [X,[Y,Z]] - (-1)^{|X||Y|}[Y,[X,Z]]$

Super $\mathfrak{gl}$

We write $\mathrm{End}(\mathbb{C}^{m|n})$ as $\mathfrak{gl}(m|n)$. As matrices: $\left(\begin{array}{c|c} A_{m\times m} & B_{m\times n} \\ \hline C_{n\times m} & D_{n\times n}\end{array}\right)$

$\mathfrak{gl}_{\bar{0}}$: $\left(\begin{array}{c|c} A_{m\times m} & 0_{m\times n} \\ \hline 0_{n\times m} & D_{n\times n}\end{array}\right)$ Preserves the parity of $v \in \mathbb{C}^{m|n}$ as a linear map.

$\mathfrak{gl}_{\bar{1}}$: $\left(\begin{array}{c|c} 0_{m\times m} & B_{m\times n} \\ \hline C_{n\times m} & 0_{n\times n}\end{array}\right)$ Reverses the parity of $v \in \mathbb{C}^{m|n}$ as a linear map.

The superbracket is the supercommutator $[X, Y] := XY - (-1)^{|X||Y|}YX$.

Bad news:

No Weyl's theorem on complete reducibility; Borels are not conjugates; the underlying geometry isn't as "straightforward"...

Hermitian Symmetric Superpairs

Now let's introduce the super twins for the superized problem.

First, let $(\mathfrak{g}, \mathfrak{k})$ be a pair of Lie superalgebras. If there is an element J in the center of $\mathfrak{k}$ whose adjoint action gives the $(-1, 0, 1)$-eigenspace decomposition and grading

$$\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+,$$

with $\mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{p}^+$, then we say $(\mathfrak{g}, \mathfrak{k})$ is a *Hermitian symmetric superpair*.

This is our superization of the algebraic data attached to the usual Hermitian symmetric space G/K.

The $\mathfrak{gl}$-pair

From now on, we set $(\mathfrak{g}, \mathfrak{k})$ where

$$\mathfrak{g} = \mathfrak{gl}(2p|2q), \mathfrak{k} = \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q)$$

with the following embedding of $\mathfrak{k}$ into $\mathfrak{g}$.

$$\left(\left(\begin{array}{c|c} A^{(1)}_{p\times p} & B^{(1)}_{p\times q} \\ \hline C^{(1)}_{q\times p} & D^{(1)}_{q\times q} \end{array} \right), \left(\begin{array}{c|c} A^{(2)}_{p\times p} & B^{(2)}_{p\times q} \\ \hline C^{(2)}_{q\times p} & D^{(2)}_{q\times q} \end{array} \right) \right) \mapsto \left(\begin{array}{cc|cc} A^{(1)}_{p\times p} & 0_{p\times p} & B^{(1)}_{p\times q} & 0_{p\times q} \\ 0_{p\times p} & A^{(2)}_{p\times p} & 0_{p\times q} & B^{(2)}_{p\times q} \\ \hline C^{(1)}_{q\times p} & 0_{q\times p} & D^{(1)}_{q\times q} & 0_{q\times q} \\ 0_{q\times p} & C^{(2)}_{q\times p} & 0_{q\times q} & D^{(2)}_{q\times q} \end{array} \right) \tag{1}$$

Here $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) consists of matrices with non-zero entries only in the upper right (resp. bottom left) sub-blocks in each of the four blocks.

The $\mathfrak{gl}$-pair

The involution θ can be given by $X \mapsto \mathbb{I} X \mathbb{I}$ with

$$\mathbb{I} = \left(\begin{array}{cc|cc} I_{r\times r} & & & \\ & -I_{p\times p} & & \\ \hline & & I_{s\times s} & \\ & & & -I_{q\times q} \end{array}\right)$$

An important observation is that $\mathbb{I}$ is central in $\mathfrak{k}$ and the above decomposition is the eigenspace decomposition with respect to $J := \frac{1}{2}\mathbb{I}$ which also gives the short grading.

Cheng–Wang Decomposition and Super Shimura Operators

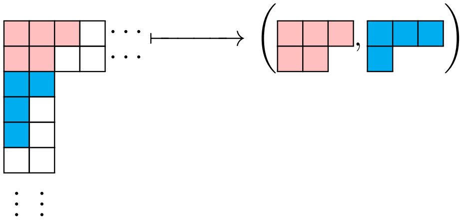
Recall we need the Schmid decomposition to define the Shimura operators D_μ. In [CW01], Cheng and Wang proved a super analog of it.

Given a partition λ, let $|\lambda|$ denote the size of λ, and λ' the transpose of λ. We define $\mathscr{H}(p,q) = \{\lambda : \lambda_{p+1} \leq q\}$ (*(p,q)-hooks*) in which we let

$$\lambda^\natural = \left(\lambda_1, \ldots, \lambda_p, \left\langle \lambda'_1 - p \right\rangle, \ldots, \left\langle \lambda'_q - p \right\rangle\right) \tag{2}$$

where $\langle x \rangle := \max\{x, 0\}$ for $x \in \mathbb{Z}$.

For example, consider $(3,2,2,1,1) \in \mathscr{H}(2,2)$, then $(3,2,2,1,1)^\natural = (3,2,3,1)$



Cheng–Wang Decomposition and Super Shimura Operators

Proposition (Cheng–Wang Decomposition, [CW01])

The symmetric algebras $\mathfrak{S}(\mathfrak{p}^+)$ and $\mathfrak{S}(\mathfrak{p}^-)$ are completely reducible and multiplicity free. Specifically,

$$\mathfrak{S}(\mathfrak{p}^+) = \bigoplus_{\lambda \in \mathscr{H}(p,q)} W\left(\lambda^\natural\right), \ \mathfrak{S}(\mathfrak{p}^-) = \bigoplus_{\lambda \in \mathscr{H}(p,q)} W^*\left(\lambda^\natural\right) \tag{3}$$

Definition

We then define D_λ as the image of the following composition of maps

$$\begin{array}{c} \mathrm{End}_{\mathfrak{k}}\left(W\left(\lambda^\natural\right)\right) \cong \left(W^*\left(\lambda^\natural\right) \otimes W\left(\lambda^\natural\right)\right)^{\mathfrak{k}} \hookrightarrow \left(\mathfrak{U}\left(\mathfrak{p}^-\right) \otimes \mathfrak{U}\left(\mathfrak{p}^+\right)\right)^{\mathfrak{k}} \to \mathfrak{U}\left(\mathfrak{g}\right)^{\mathfrak{k}} \\ 1 \longmapsto D_\lambda \end{array}$$

The element D_λ is called the *super Shimura operator associated with* λ.

Iwasawa Decomposition and Harish-Chandra Homomorphism

Goal: relate the Harish-Chandra image of D_μ with some polynomial.
Let us discuss the Harish-Chandra homomorphism first. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m(\alpha)\alpha$ be the *half sum of positive roots*. Here $m(\alpha) := \operatorname{sdim}\mathfrak{g}_\alpha = \dim(\mathfrak{g}_\alpha)_{\bar{0}} - \dim(\mathfrak{g}_\alpha)_{\bar{1}}$ is the multiplicity of α.

- Iwasawa Decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (also the opposite: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^-$)
  - $\mathfrak{a}$: even Cartan subspace, $\mathfrak{n} := \bigoplus_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$: the nilpotent subalgebra for some positive system of Σ
  - Does NOT always exist. Need: $(\mathfrak{g}, \mathfrak{k})$ a reductive symmetric superpair of *even type*.
  - Can be explicitly written for our pair.
- Harish-Chandra projection. The PBW theorem allows us to write $\mathfrak{U}(\mathfrak{g}) = (\mathfrak{U}\mathfrak{k} + \mathfrak{n}^-\mathfrak{U}) \oplus \mathfrak{S}(\mathfrak{a})$. Define π to be the projection onto $\mathfrak{S}(\mathfrak{a})$.
  - π depends on a choice of positivity.
- Harish-Chandra homomorphism. $\Gamma(D)(\lambda) = \pi(D)(\lambda - \rho)$.
  - The minus sign is due to the opposite $\mathfrak{n}^-$.

Iwasawa Decomposition and Harish-Chandra Homomorphism

In [All12], Alldridge introduced certain subalgebra $J_\alpha \subseteq \mathfrak{S}(\mathfrak{a})$, and described the map Γ in terms of its kernel and image. We specialize to our pair as follows:

Theorem (Alldridge, [All12])

We have $\ker \Gamma = \mathfrak{k}\mathfrak{U}(\mathfrak{g}) \cap \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}}$ *and*

$$\operatorname{Im}\Gamma = \mathfrak{S}\left(\mathfrak{a}\right)^{W_0} \bigcap_{\alpha \in {}_0\Sigma} J_\alpha .$$

All these J_α can be written out explicitly, which is vital for computations. Wait, what are W_0 and ${}_0\Sigma$??

A Digression

Restricted Root System $\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a})$. We can also mimic the construction of an *even* Cartan subspace $\mathfrak{a} \subseteq \mathfrak{p}$ (using the Harish-Chandra roots) consisting matrices of the following form:

$$H := \left(\begin{array}{cc|cc} 0 & \operatorname{diag}(\mathbf{a}^{\mathrm{B}} i) & \multicolumn{2}{c}{0} \\ \operatorname{diag}(-\mathbf{a}^{\mathrm{B}} i) & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & \operatorname{diag}(\mathbf{a}^{\mathrm{F}} i) \\ & & \operatorname{diag}(-\mathbf{a}^{\mathrm{F}} i) & 0 \end{array}\right)$$

Here B/F refer to Boson and Fermion respectively, as a convention borrowed from physics. Then it can be verified that the restricted root system $\Sigma = \Sigma_{\overline{0}} \sqcup \Sigma_{\overline{1}}$ is of *Type C* whose *even* roots constitute a root system of Type $C(p) \sqcup C(q)$. This gives us W_0 which is of course of Type BC.

But there are *odd roots* too! It turns out that they are *isotropic* (norm 0) too. That's what ${}_0\Sigma$ is.

Interpolation Polynomials

Let $\{\epsilon_1, \ldots, \epsilon_p, \delta_1, \ldots, \delta_q\}$ be the standard basis of $\mathbb{C}^{p|q}$.

- 5 parameters: $\mathsf{p}, \mathsf{q}, \mathsf{k}, \mathsf{r}, \mathsf{s}$ and 2 relations: $\mathsf{p} = \mathsf{kr}, 2\mathsf{q} + 1 = \mathsf{k}(2\mathsf{s} + 1)$
- inner product:$(\epsilon_i, \epsilon_j) = \mathsf{k}^{-1}(\delta_i, \delta_j) = \delta_{i,j}, \ (\epsilon_i, \delta_j) = 0$

Set $\mathsf{h} := -\mathsf{k}p - q - \frac{1}{2}\mathsf{p} - \mathsf{q}$. Let $\varrho = (\varrho_1^{\mathrm{B}}, \ldots, \varrho_p^{\mathrm{B}}, \varrho_1^{\mathrm{F}}, \ldots, \varrho_q^{\mathrm{F}})$ where

$$\varrho_i^{\mathrm{B}} := -(\mathsf{h} + \mathsf{k}i), \ \varrho_j^{\mathrm{F}} := -\mathsf{k}^{-1}\left(\mathsf{h} + \mathsf{k}/2 - 1/2 + j + \mathsf{k}p\right).$$

Definition (Polynomials of Type BC Shifted-supersymmetry)

Define Λ_Σ^ϱ as the subalgebra of polynomials $f \in P_{p,q} := \mathbb{C}[z_i, w_j]_{i,j=1}^{p,q}$ which

1. are symmetric separately in $(z_i - \varrho_i^{\mathrm{B}})$ and $(w_j - \varrho_j^{\mathrm{F}})$, and invariant under their sign changes; and
2. satisfy $f(X + \alpha) = f(X)$ when X is in the hyperplane defined by $(X - \varrho, \alpha) + \frac{1}{2}(\alpha, \alpha) = 0$ for all $\alpha = \epsilon_i \pm \delta_j$.

Interpolation Polynomials

What this ring of "supersymmetric" polynomials *really* depends on is a deformed

Now in this ring , we can finally define our protagonist — the Sergeev and Veselov polynomial!

Theorem (Sergeev and Veselov, [SV09])

For each $\mu \in \mathscr{H}(p,q)$, there exists a unique polynomial $I_\mu \in \Lambda^{\varrho}_{\Sigma}$ of degree $2|\mu|$ such that

$$I_\mu(\lambda^\natural; \mathsf{k}, \mathsf{h}) = 0$$

*for any $\lambda \not\supseteq \mu$ (*the **vanishing condition***) and normalized with*

$$I_\mu(\mu^\natural; \mathsf{k}, \mathsf{h}) = \prod_{(i,j)\in\mu} \left(\mu_i - j - \mathsf{k}(\mu'_j - i) + 1\right)\left(\mu_i + j + \mathsf{k}(\mu'_j + i) + 2\mathsf{h} - 1\right).$$

Interpolation Polynomials

Since the restricted root system $\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a})$ is specified, we know exactly $\mathsf{k} = -1$ and $\mathsf{h} = p - q + \frac{1}{2}$ by computations. Also, $\varrho = -\frac{1}{2}\rho$.

Definition

Define Λ_Σ^ϱ as the subalgebra of polynomials $f \in P_{p,q} := \mathbb{C}[z_i, w_j]_{i,j=1}^{p,q}$ which

1. are symmetric separately in $(z_i - \varrho_i^{\mathrm{B}})$ and $(w_j - \varrho_j^{\mathrm{F}})$, and invariant under their sign changes; and
2. satisfy $f(X + \alpha) = f(X)$ when X is in the hyperplane defined by $(X - \varrho, \alpha) + \frac{1}{2}(\alpha, \alpha) = 0$ for all $\alpha = \epsilon_i \pm \delta_j$.

Definition (Specified to $(\mathfrak{gl}(2p|2q), \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q))$)

$\Lambda_\Sigma^{-\frac{1}{2}\rho}$ is the subalgebra of polynomials $f \in P_{p,q} := \mathbb{C}[z_i, w_j]_{i,j=1}^{p,q}$ which

1. are symmetric separately in $(z_i + (p - i) + 1/2 - q)$ and $(w_j + (q - i) + 1/2)$, and invariant under their sign changes; and

Results

Recall $\operatorname{Im}\Gamma = \mathfrak{S}(\mathfrak{a})^{W_0} \bigcap_{\alpha \in {}_0\Sigma} J_\alpha$. *Lo and behold!*

Proposition ([Zhu])

The algebra $\operatorname{Im}\Gamma$ *consists precisely of the symmetric polynomials on* $\mathfrak{a}^*$ *with Type BC supersymmetry property.*

Can be reformulated as

Proposition (Weyl Groupoid Formulation)

$$\operatorname{Im}\Gamma = \mathfrak{S}(\mathfrak{a})^{\mathfrak{W}} \cong \mathfrak{P}(\mathfrak{a}^*)^{\mathfrak{W}}$$

$\mathfrak{W}$ represents the Weyl groupoid associated with the restricted root system, which acts in a way such that the supersymmetry is captured.

Results

Finally, $\Gamma : \mathfrak{U}^{\mathfrak{k}} \to \mathfrak{P}(\mathfrak{a}^*)^{\mathfrak{W}}$ is legal. The following proposition makes sure that the symmetry of $\operatorname{Im}\Gamma$ and that of $\Lambda_{\Sigma}^{-\frac{1}{2}\rho}$ match up (just a change of variable/ρ-shift):

Proposition ([Zhu])

The ring $\Lambda_{\Sigma}^{-\frac{1}{2}\rho}$ and $\mathfrak{P}(\mathfrak{a}^*)^{\mathfrak{W}} = \operatorname{Im}\Gamma$ are isomorphic via an isomorphism τ.

Theorem ([Zhu])

Assuming a conjecture, the Harish-Chandra image of the super Shimura operator associate with μ, $\Gamma(D_\mu)$, is equal to some non-zero multiple of $\tau(I_\mu)$.

Main Ideas

Before we talk about the conjecture...

Recall we introduced the $\mathfrak{k}$-irreducible modules $W(\lambda)$ in the Schmid (Cheng–Wang) decomposition. In [SZ19], a family of $\mathfrak{g}$-modules $V(\lambda)$ is considered in the proof.

Key observations

1. D_μ acts on the *spherical* vector in $V(\lambda)$ by the scalar $\Gamma(\lambda + \rho)$.
   - Guaranteed by the Cartan–Helgason Theorem.
2. A branching statement: if $\lambda \not\supseteq \mu$, then $\mathrm{Hom}_{\mathfrak{k}}(W(\mu), V(\lambda))$ is ZERO.
   - A highest weight theoretical proof.
3. The vanishing condition follows from the branching statement!
   - ...with the help of some more rep theory machinery.

The spherical vector is crucial. It is not known if $V(\lambda^\natural)$ is always spherical.

Spherical Representations

Let $(\mathfrak{g}, \mathfrak{k})$ be a pair of Lie (super)algebras associated with (G, K). We say a $\mathfrak{g}$-module V is *spherical* if $V^{\mathfrak{k}} := \{v \in V : X.v = 0 \text{ for all } X \in \mathfrak{k}\}$ is non-zero. What are those $V(\lambda)$, or $V(\lambda^{\natural})$ in the super scenario?

Answer

$V(\lambda)$ is defined to be the *irreducible module of highest weight* $\sum \lambda_i \gamma_i$. This makes sense (see Background). This weight is highest w.r.t the Borel subalgebra of $\mathfrak{g}$ extended from the one of $\mathfrak{k}$ by $\mathfrak{p}^+$.

Spherical Representations

For our $(\mathfrak{gl}(2p|2q), \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q))$:

Conjecture

Every irreducible $\mathfrak{g}$-module $V\left(\lambda^{\natural}\right)$ for $\lambda \in \mathscr{H}(p,q)$ is spherical.

We proved a partial result:

Theorem ([Zhu])

For $p = q = 1$, all the irreducible $\mathfrak{g}$-modules $V\left(\lambda^{\natural}\right)$ are spherical.

Spherical Representations

Why don't we need any conjecture in the classical picture?
Recall we have two Cartans. The "standard" $\mathfrak{t}$, and the $\mathfrak{h}$ containing $\mathfrak{a}$, a Cartan subspace in $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
In the classical picture:

Theorem (Cartan–Helgason Theorem, [Hel00])

Let V be a irreducible $\mathfrak{g}$-module of highest weight $\lambda \in \mathfrak{h}$. Then V is spherical if and only if

1. $\lambda_\alpha := \frac{(\lambda|_\mathfrak{a}, \alpha)}{(\alpha, \alpha)} \in \mathbb{N}$ *for* $\alpha \in \Sigma^+$; *and*
2. $\lambda |_{\mathfrak{h} \cap \mathfrak{k}} = 0$.

Spherical Representations

In the super picture:

Theorem (Alldridge, [AS15])

Let V be a $\mathfrak{g}$-module of highest weight $\lambda \in \mathfrak{h}$. Then V is spherical if

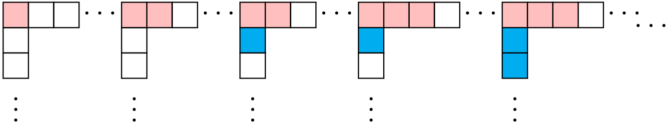
1. $\lambda_\alpha := \frac{(\lambda|_{\mathfrak{a}},\alpha)}{(\alpha,\alpha)} \in \mathbb{N}$ *for* $\alpha \in \Sigma^+_{\bar{0}}$;
2. $\lambda\,|_{\mathfrak{h}\cap\mathfrak{k}} = 0$; *and*
3. λ *is* high enough.

1. It is sufficient but not necessary;
2. The trivial rep ($\lambda = 0$) is spherical, but is not high enough;
3. The high enough condition is a purely odd condition. Technical. Involves root multiplicities.

For $\lambda \in \mathscr{H}(p,q)$, this is NOT enough to deduce that each $V(\lambda^\natural)$ is spherical!

Spherical Representations

An $(p, q) = (1, 1)$ example.
We proved: $V(\lambda^\natural)$ is spherical for all $\lambda \in \mathscr{H}(1, 1)$.
What the above theorem implies:



The arm must be longer than the leg.

Spherical Representations

Theorem ([Zhu])

For $p = q = 1$, all the irreducible $\mathfrak{g}$-modules $V\left(\lambda^\natural\right)$ are spherical.

Sketch of the proof.

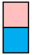
Let V be a $\mathfrak{g}$-module with the maximal submodule M. A non-zero vector $v \in V$ is said to be *quasi-spherical* if $\mathfrak{k}.v \subseteq M$ and $\mathfrak{U}(\mathfrak{g}).v = V$, i.e. cyclic. A $\mathfrak{g}$-module is called *quasi-spherical* if it has such a quasi-spherical vector.
Key ingredient: Kac modules $K(\check{\lambda}) := \mathrm{Ind}_{\mathfrak{g}_{\bar{0}}+\mathfrak{g}_1}^{\mathfrak{g}} \check{W}(\check{\lambda}) \cong \bigwedge(\mathfrak{g}_{-1}) \otimes \check{W}(\check{\lambda})$. where $\check{W}$ is the irreducible $\mathfrak{g}_{\bar{0}}$-module with highest weight $\check{\lambda}$, and we extend the action of $\mathfrak{g}_{\bar{0}}$ trivially to $\mathfrak{r} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_1$.
This $\check{\lambda}$ is a result of changing the Borel of $\mathfrak{g}$ to the *distinguished* one. This step is non-trivial in the super scenario. This is what essentially stops us from generalizing p, q.

Spherical Representations

Sketch of the proof (cont.d).

Punchline 1: $V(\lambda^\natural)$ is the irreducible quotient of $K(\check{\lambda})$.
Punchline 2: $K(\check{\lambda})$ is indeed quasi-spherical.
Only two possibilities for $\lambda = (a, 1^b) \in \mathscr{H}(1,1)$ are present. We show when $b \neq a-1$, $K(\check{\lambda})$ has a spherical vector that descends to $V\left(\lambda^\natural\right)$. For $b = a-1$, by studying "degree 2" operators in $\mathfrak{U}(\mathfrak{g})$, we prove that $K(\check{\lambda})$ is indeed quasi-spherical. The proof is computational. □

An Example

We explain why the high enough condition for sphericity is insufficient. Let $p = q = 1$ so $\rho = (-1, 1)$. Let $\mu = (2) =$  $\in \mathscr{H}(1,1)$.

The vanishing condition of $\tau(I_\mu)$ says it is zero for any λ not containing (2), i.e. the partitions (1^n) for $n \in \mathbb{N}$ (one-column partitions).
Since "arm > leg", the only partition above guaranteed to give a spherical $V\left(\lambda^\natural\right)$ is $\lambda = (1)$. Plug in the weights and we have $\Gamma(D_\mu)((1,1)) = 0$.
Normalize $\Gamma(D_\mu) \in P[x,y]$ so that the resulting polynomial f has leading coefficient 1. Then as a Type BC supersymmetric polynomial in two variable of degree 4, we may assume $f(x,y) = (x^2 - y^2)(x^2 + ay^2 + b)$.
But $f(1,1) = 0$ does not solve a and b!
In fact, in this case we have $f(x,y) = (x^2 - y^2)(x^2 - 1)$ which indeed vanishes at $(1, 2n-1)$ for all $\lambda = (1^n) \not\supseteq \mu = (2)$.

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