Supersymmetric Shimura operators and interpolation polynomials

Joint work with Siddhartha Sahi

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Algebra Seminar
September 30, 2024
arxiv.org/abs/2212.09249
arxiv.org/abs/2312.08661

Structure

- Background
  - Origin
  - Some Lie theory
  - Shimura operators
- Symmetric polynomials
  - Okounkov polynomials
  - Sergeev–Veselov polynomials
- Results and scope [Zhu22, SZ23]

Punch lines

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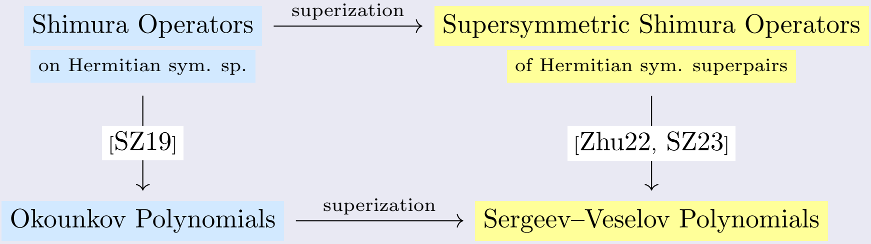
Images: symmetric polynomials; good maps: Harish-Chandra isom.
We solved the Type A super analog:



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7. The theory of symmetric functions gives answers to Shimura's problem.

Some Lie stuff

Lie groups $G, K, \ldots \xrightarrow{\text{c, noncpt/cpt., s.s., etc}}$ Lie algebras $\mathfrak{g}, \mathfrak{k}, \ldots$
We will: look at the complex(ified) Lie (super)algebras; consistently use $\mathfrak{Frakturs}$ to denote Lie algebras and subspaces therein.

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1. Tangent space at e_G;
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   2.1 $[X, X] = 0$ (skew symmetry, char $\neq 2$)
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$\mathfrak{gl}(V)$: all endomorphisms on V, Lie bracket: $[f, g] = fg - gf$

$\mathfrak{g}$-module/representation

$(\mathfrak{g} \xrightarrow{\pi} \mathfrak{gl}(V), V)$: $\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$

Often write $X.v$ for $\pi(X)(v)$. So $[X, Y].v = X.(Y.v) - Y.(X.v)$.

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Finite dimensional ones $\leftrightarrow$ Dominant & integral λ (like a partition!)

Example

$\mathfrak{gl}(2)$ $\quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $[X, Y] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: H$.
Cartan $\mathfrak{t}:=$ diag. matrices; Borel $\mathfrak{b} := \mathfrak{t} \oplus \mathbb{C}X$; Let ϵ_i be the coordinate of the i-th entry on the diagonal, then $(3, -1) = 3\epsilon_1 - \epsilon_2$ gives a 5D irrep while $(-1, 3)$ or $(3/2, 1)$ do not ($(3/2, 1/2)$ does).

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Can't multiply on $\mathfrak{g}$, unlike in $\mathfrak{gl}(V)$ (multiplication of matrices).
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Theorem (Poicaré–Birkhoff–Witt Theorem for basis)

Let $\{X_1, \ldots, X_n\}$ *be an ordered basis for* $\mathfrak{g}$*. Then* $\{X_{i_1} \cdots X_{i_k} : i_1 \leq \cdots \leq i_k\}$ *is a basis for* $\mathfrak{U}(\mathfrak{g})$*, and* $X_iX_j - X_jX_i = [X_i, X_j] \in \mathfrak{g}$*. (In other words, this tells us how to multiply things and sort them, and the associated graded algebra* $\mathrm{gr}(\mathfrak{U}(\mathfrak{g})) \cong \mathfrak{S}(\mathfrak{g})$*.)*

Universality: $\mathfrak{g}$-mod $= \mathfrak{U}(\mathfrak{g})$-mod.
Special case: for abelian $\mathfrak{g}$, $\mathfrak{U}(\mathfrak{g}) = \mathfrak{S}(\mathfrak{g})$.

In context

Let $X := G/K$ be as above. Change the point of view to Lie algebras.

1. $(\mathfrak{g}, \mathfrak{k})$: Harish-Chandra decomposition $[(-1, 0, 1)$-grading$]$

$$\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+ (= \mathfrak{k} \oplus \mathfrak{p})$$

$$[\mathfrak{k}, \mathfrak{p}^\pm] = \mathfrak{p}^\pm, [\mathfrak{p}^+, \mathfrak{p}^+] = [\mathfrak{p}^-, \mathfrak{p}^-] = 0, [\mathfrak{p}^+, \mathfrak{p}^-] = \mathfrak{k}$$

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E.g. $(\mathfrak{gl}(2n), \mathfrak{gl}(n) \oplus \mathfrak{gl}(n))$ (The center doesn't matter here)
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2. $\mathfrak{D} = \mathfrak{D}(X)$: space of invariant differential operators on X.
$\mathfrak{U} := \mathfrak{U}(\mathfrak{g})$; $\mathfrak{U}^{\mathfrak{k}} := \mathfrak{Z}_{\mathfrak{k}}(\mathfrak{U})$; $(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}} := \mathfrak{U}\mathfrak{k} \cap \mathfrak{U}^{\mathfrak{k}}$.
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5. $\gamma^0 : \mathfrak{D} \to \Lambda \subseteq \mathfrak{P}(\mathfrak{a}^*)$: the Harish-Chandra isomorphism. Good map! Essentially a symmetry-preserving projection, shifted by ρ (determined by $\mathfrak{b}$)

Schmid decomposition and Shimura operators

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Here we choose a form to identify $\mathfrak{p}^-$ *with* $(\mathfrak{p}^+)^*$.

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Shimura Operators

$$\mathrm{End}_{\mathfrak{k}}(W_\lambda) \cong (W_\lambda^* \otimes W_\lambda)^{\mathfrak{k}} \hookrightarrow \left(\mathfrak{S}\left(\mathfrak{p}^-\right) \otimes \mathfrak{S}\left(\mathfrak{p}^+\right)\right)^{\mathfrak{k}} \to \mathfrak{U}^{\mathfrak{k}} \to \mathfrak{D}$$
$$1 \longmapsto D_\lambda \mapsto \mathscr{D}_\lambda$$

Call $\mathscr{D}_\lambda$ the *Shimura operator associated with* λ.

A **basis** parameterized by **partitions**! Nice images of 1's.

Lie superalgebras

General Principle of Superization ($\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$)

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Definition

A *vector superspace* V is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. A vector $v \in V_{\bar{0}}$ (resp. $V_{\bar{1}}$) is said to be *even* (resp. *odd*) and we set $|v| = 0$ (resp. 1). We let $\mathbb{C}^{m|n}$ denotes the vector superspace with even subspace $\mathbb{C}^m$ and odd subspace $\mathbb{C}^n$.

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A *Lie superalgebra* is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ (*Lie superbracket*) satisfying

1. $[X, Y] = -(-1)^{|X||Y|}[Y, X]$
2. $[[X, Y], Z] = [X, [Y, Z]] - (-1)^{|X||Y|}[Y, [X, Z]]$

Or: the "Lie" object in the category `SVect`.

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"Bad" news for LSA

No Weyl's theorem on complete reducibility; Borels are not conjugates; *isotropic* (restricted) roots...

Lie superalgebras

Fix $\mathfrak{g} = \mathfrak{gl}(2p|2q)$ and $\mathfrak{k} = \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q)$. Embed $\mathfrak{k}$ into $\mathfrak{g}$:

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Why this pair only?

1. Some argument depends on coordinates. This should be generalized.
2. Irreps highest weights are easy to compute.

Supersymmetric Shimura operators

$\mathscr{H} = \mathscr{H}(p,q) := \{\lambda : \lambda_{p+1} \leq q\}$ (hook partitions).
$\mathscr{H}^d := \{\lambda \in \mathscr{H} : |\lambda| = d\}$.
Super analog of the Schmid decomposition: ([CW01, SSS20])

$$\mathfrak{S}^d(\mathfrak{p}^+) = \bigoplus_{\lambda \in \mathscr{H}^d} W_\lambda,\ \mathfrak{S}^d(\mathfrak{p}^-) = \bigoplus_{\lambda \in \mathscr{H}^d} W_\lambda^*.$$

W_λ are of Type M, and $\dim \mathrm{End}_{\mathfrak{k}}(W_\lambda) = 1$. (Super version of Schur's Lemma may include the parity twist.) Set $\mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$.

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And yes, the super Harish-Chandra isomorphism $\gamma^0 : \mathfrak{D} \to \Lambda \subseteq \mathfrak{P}(\mathfrak{a}^*)$ exists, but not well-understood.

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We work with n variables. Denote $\prod x_i^{a_i}$ by x^a for $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. The usual ones:

1. $m_\lambda := \sum_{\text{all perm } \alpha \text{ of } \lambda} x^\alpha$ [e.g. $m_{(2,1,1)}(x_1, x_2, x_3) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$]
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$$s_\lambda = \sum_T x^T$$

T: Semistandard Young tableau of λ. Fill in $\{1, \ldots, n\}$ in λ so they strictly increase along columns, weakly increase along rows.

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This is *Type A* symmetry: S_n is the Weyl group of Type A.
For Type BC: include the sign changes. Weyl group $= S_n \ltimes \mathbb{Z}_2^n$.

Okounkov polynomials

Λ is the ring of Type BC symmetric polynomials.
$\Lambda := \mathbb{C}[x_1, \ldots, x_n]^{S_n \ltimes \mathbb{Z}_2^n}$ (ring of even symmetric polynomials)
$\rho := (\rho_1, \ldots, \rho_n)$, $\rho_i := \tau(n-i) + \alpha$. τ, α: parameters.

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Theorem-Definition [Oko98, OO06], c.f. [SZ19]

The Okounkov polynomial $P_\mu(x; \tau, \alpha)$ is the unique polynomial in Λ satisfying

1. $\deg P_\mu = 2|\mu|$;
2. $P_\mu(\lambda + \rho) = 0$ for $\lambda \not\supseteq \mu$ [**vanishing properties**];
3. Some normalization condition.

Has a similar tableau formula $\sum_T \varphi_T(\tau) \prod_{s \in \mu} \left(x_{T(s)}^2 - \#(s, \tau, \alpha) \right)$ where $\varphi_T(\tau)$ is a coefficient/weight associated to each T (reversed tableau of μ).

ρ can be specialized to the half sum of positive roots for a restricted root system of Type BC. [The case for Hermitian X]

Examples

Consider the one-row partition (l) for $l \in \mathbb{N}$.
Then $\rho = \alpha$. We have [SZ19]:

$$P_{(l)}(x;\tau,\alpha) = \prod_{k=0}^{l-1}(x^2 - (k+\alpha)^2).$$

- Even symmetric;
- Degree $2l$;
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For (l^n), we have

$$P_{(l^n)}(x;\tau,\alpha) = \prod_{i=0}^{l-1}\prod_{j=1}^{n}(x_j^2 - (i+\alpha)^2).$$

Classical result

Recall the H-C isomorphism $\gamma^0 : \mathfrak{D} \to \Lambda$. It captures/transports the Type BC symmetry (restricted root system $\to$ polynomials).

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We have $\gamma^0(\mathscr{D}_\lambda) = k_\lambda P_\lambda$ *for some explicit* $k_\lambda \neq 0$.

Let V_μ be the irreducible $\mathfrak{g}$-module (of the same highest weight as W_μ). Then by the Cartan–Helgason Theorem, V_μ has a spherical vector $v^{\mathfrak{k}}$, i.e. $\mathfrak{k}.v^{\mathfrak{k}} = 0$. $D_\lambda \in \mathfrak{U}^{\mathfrak{k}}$ ($\mathscr{D}_\lambda \in \mathfrak{D}$) acts on $v^{\mathfrak{k}}$ as $\gamma^0(\mathscr{D}_\lambda)(\mu + \rho)$, hence the word spectrum/eigenvalue!

Supersymmetric polynomials

For polynomials, supersymmetry = symmetry + translational invariance.

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More precisely, in the Type BC situation:

Definition

Let $\{\epsilon_i\}_{i=1}^p \cup \{\delta_j\}_{j=1}^q$ be the standard basis for $V = \mathbb{C}^{p+q}$. Denote the coordinate functions for this basis as x_i and y_j. A polynomial $f \in \mathfrak{P}(V) = \mathbb{C}[x_i, y_j]$ is said to be *even supersymmetric* if :

(i) f is symmetric in x_i and y_j separately and invariant under sign changes of x_i and y_j;

(ii) $f(X + \epsilon_i - \delta_j) = f(X)$ if $x_i + y_j = 0$ for $i = 1, \ldots, p$, $j = 1, \ldots, q$.

We denote the subring of even supersymmetric polynomials as $\Lambda^0(V)$.

!! (i) $\iff$ invariance under $(S_p \ltimes \mathbb{Z}_2^n) \times (S_q \ltimes \mathbb{Z}_2^n)$ but (ii) is no longer group invariance (not even linear).

In fact, one may use a suitable Weyl *groupoid* action to capture (i & ii).

Sergeev–Veselov Polynomials

In [SZ23], we proved that $\operatorname{Im}\gamma^0$ is exactly $\Lambda^0(\mathfrak{a}^*)$, the ring of even supersymmetric polynomials on $\mathfrak{a}^*$, previously proved also in [Zhu22].

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Proposition-Definition [SV09]

For each $\mu \in \mathscr{H}$, there is a unique polynomial $J_\mu \in \Lambda^0$ of degree $2|\mu|$ s.t.

$$J_\mu(\overline{\lambda} + \rho) = 0, \quad \text{for all } \lambda \not\supseteq \mu,\ \lambda \in \mathscr{H}$$

and that $J_\mu(\overline{\mu} + \rho) =$ explicit non-zero number. (Has a tableau formula too!)

Here $\overline{\lambda}$ is some choice of coordinates (*Frobenius coordinates*). ρ is the Weyl vector, the half sum of the positive restricted roots.

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E.g. $p = q = 1$, for the restricted root system, $\rho = (-1, 1)$.

1. $\mu = (1)$, $\lambda = \varnothing$, and $\overline{\lambda} + \rho = (-1, 1)$, $J_{(1)} \propto x^2 - y^2$.

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2. $\mu = (2)$, $\lambda = (1^n)$, and $\overline{\lambda} + \rho = (1, 2n - 1)$, $J_{(2)} \propto (x^2 - y^2)(x^2 - 1)$.

Main Result

Theorem A (Sahi–Z. [SZ23])

We have $\gamma^0(\mathscr{D}_\mu) = k_\mu J_\mu$ *where* $k_\mu = (-1)^{|\mu|} \prod_{(i,j)\in\mu} \left(\mu_i - j + \mu'_j - i + 1\right)$.

Main thing: show the vanishing properties.

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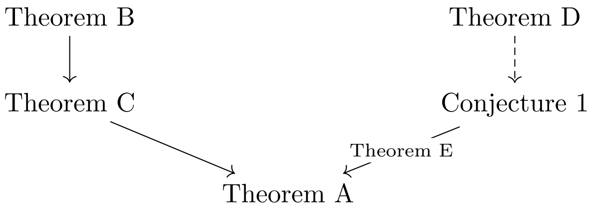
Indirect [SZ23]

Direct (partial result) [Zhu22]
(analogous to [SZ19])



Results [SZ23]

Let the center of $\mathfrak{U}$ be $\mathfrak{Z}$. Then $\mathfrak{Z} \subseteq \mathfrak{U}^{\mathfrak{k}}$ and we have $\pi : \mathfrak{Z} \hookrightarrow \mathfrak{U}^{\mathfrak{k}} \twoheadrightarrow \mathfrak{D}$.

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The map π is surjective. In particular, there exist $Z_\mu \in \mathfrak{Z}$ such that $\pi(Z_\mu) = \mathscr{D}_\mu$.

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The map π is surjective. In particular, there exist $Z_\mu \in \mathfrak{Z}$ such that $\pi(Z_\mu) = \mathscr{D}_\mu$. (So $\mathscr{D}_\mu = \pi(D_\mu)$ can be captured by some central element!)

Let $I_\lambda := \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda$ be the generalized Verma module for $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$.

Theorem C (Sahi–Z. [SZ23])

The central element Z_μ acts on I_λ by 0 when $\lambda \not\supseteq \mu$.

The point: Use rep theoretic machinery to get the vanishing properties.

Results

Theorem B

The map $\pi : \mathfrak{Z} \to \mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$ is surjective. In particular, there exist $Z_\mu \in \mathfrak{Z}$ such that $\pi(Z_\mu) = \mathscr{D}_\mu$.

$$\begin{CD} \mathfrak{Z} @>{\pi}>> \mathfrak{D} \\ @V{\gamma}V{\wr}V @V{\wr}V{\gamma^0}V \\ \Lambda(\mathfrak{h}^*) @>{\texttt{Res}}>> \Lambda^0(\mathfrak{a}^*) \end{CD}$$

$\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t}_+$: Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$.

$\gamma : \mathfrak{Z} \to \mathfrak{P}(\mathfrak{h}^*)$: the usual Harish-Chandra isomorphism defined on $\mathfrak{Z}$. *Much well understood!*

`Res`: the restriction map induced from the decomposition $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}_+$.

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Sketch of the proof of Theorem C

1. $I_\lambda = \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda \cong \mathfrak{S}(\mathfrak{p}^-) \otimes W_\lambda \cong \bigoplus \left(W_\mu^* \otimes W_\lambda\right)$ (as $\mathfrak{k}$-modules).
   Spherical: $I_\lambda^{\mathfrak{k}} \subseteq W_\lambda^* \otimes W_\lambda$ with $\dim I_\lambda^{\mathfrak{k}} = 1$.

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3. $\mathrm{Hom}_\mathfrak{k}(W_\mu, I_\lambda) = \{0\}$ for $\lambda \not\supseteq \mu$.
4. $D_\mu = \sum \xi_i \eta_i$ for $\xi_i \in W_\mu^*$ and $\eta_i \in W_\mu$. So $D_\mu . I_\lambda^\mathfrak{k} = \{0\} = \mathscr{D}_\mu . I_\lambda^\mathfrak{k}$.

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5. Z_μ also acts on $I_\lambda^{\mathfrak{k}}$ by 0. Acts by 0 on the entirety of I_λ (since $Z_\mu \in \mathfrak{Z}$)!

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The main thing is that I_λ has $\mathfrak{k}$-highest weight and is infinite dimensional. We don't know by what "polynomial" $\mathscr{D}_\mu$ acts on $I_\lambda^{\mathfrak{k}}$ directly!

Results

Theorem A

We have $\gamma^0(\mathscr{D}_\mu) = k_\mu J_\mu$ where $k_\mu = (-1)^{|\mu|}\prod_{(i,j)\in\mu}\left(\mu_i - j + \mu'_j - i + 1\right)$.

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By Theorem C, Z_μ acts by 0 on I_λ for $\lambda \not\supseteq \mu$. But $\mathfrak{Z}$ acts exactly by γ. Thus

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$$\gamma^0(\mathscr{D}_\mu)(\overline{\lambda} + \rho) = 0, \text{for all } \lambda \not\supseteq \mu,\ \lambda \in \mathscr{H}.$$

To pin down k_μ, we compare the leading term of $\gamma^0(\mathscr{D}_\mu)$ and that of J_μ, which has a tableau formula. The tableau formula of the super Jack polynomials ([SV05]) solves k_μ in the end.

Results

[Zhu22] Old method:
Consider the irreducible quotient V_λ of I_λ.

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Consider the irreducible quotient V_λ of I_λ. Fact: By [Zhu22, Theorem 4.3], V_λ is finite dimensional (*see Chapter 4 in my dissertation*). The proof is combinatorial in nature.

Conjecture 1 (Z. [Zhu22])

Every irreducible $\mathfrak{g}$-module V_λ for $\lambda \in \mathscr{H}(p,q)$ is spherical.

Theorem D (Z. [Zhu22])

Conjecture 1 *is true for $p = q = 1$.*

Theorem E (Z. [Zhu22])

Theorem A *follows from* Conjecture 1.

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Sketch of the proof of Theorem E

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Scope

	Shimura	Capelli	quadratic Capelli
☺	[SZ19]	[KS93, Sah94]	[SS19]
$\mathbb{Z}_2$	[Zhu22, SZ23]	[SSS20]	?
q	?	[LSS22]	?
$\mathbb{Z}_2, q$	?	?	?

Table: Scope of the theory. $\mathbb{Z}_2$: super. q: quantum.

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	Shimura	Capelli	quadratic Capelli
$\ddot{\smile}$	[SZ19]	[KS93, Sah94]	[SS19]
$\mathbb{Z}_2$	[Zhu22, SZ23]	[SSS20]	?
q	?	[LSS22]	?
$\mathbb{Z}_2, q$	?	?	?

Table: Scope of the theory. $\mathbb{Z}_2$: super. q: quantum.

1. Capelli identity – Capelli operators – Jordan algebras – Type A interpolation polynomials
2. Shimura eigenvalue problem – Shimura operators – Jordan algebras – Type BC interpolation polynomials

Scope

	Shimura	Capelli	quadratic Capelli
:)	[SZ19]	[KS93, Sah94]	[SS19]
$\mathbb{Z}_2$	[Zhu22, SZ23]	[SSS20]	?
q	?	[LSS22]	?
$\mathbb{Z}_2, q$	?	?	?

Table: Scope of the theory. $\mathbb{Z}_2$: super. q: quantum.

1. Capelli identity – Capelli operators – Jordan algebras – Type A interpolation polynomials
2. Shimura eigenvalue problem – Shimura operators – Jordan algebras – Type BC interpolation polynomials
3. Really similar construction. Act on different objects.

Thank you!

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