

Super Krawtchouk Polynomials via Lie Superalgebras

Joint work with Plamen Iliev

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AMS SECTIONAL MEETING, SAVANNAH, SPECIAL SESSION
ON NONCOMMUTATIVE ALGEBRAS, QUANTUM GROUPS, AND RELATED TOPICS
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- They are orthogonal. *Using rep theory.*
- They have (fermionic) probability interpretation. *“Quantum” stuff.*

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- Griffiths also proposed a multivariate version
- Iliev gave a Lie theoretic interpretation [Ili12]

An example

Fix $N \in \mathbb{Z}_{\geq 0}$. For $m, x \in \{0, 1, \dots, N\} =: [N]$ set

$$(1+z)^{N-x}(1-z)^x = \sum_{m=0}^N \binom{N}{m} \mathcal{P}_{\bar{0}}(x, m; N, 1/2) z^m.$$

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($\bar{0}$ means even). Orthogonality

$$\sum_{x=0}^N \binom{N}{x} \mathcal{P}_{\bar{0}}(x, m) \mathcal{P}_{\bar{0}}(x, k) = \delta_{m,k} \frac{2^N}{\binom{N}{m}}.$$

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$$\mathcal{P}_{\bar{0}}(x, 0) = 1, \quad \mathcal{P}_{\bar{0}}(x, 1) = \frac{1}{N}(-x + 2N), \quad \mathcal{P}_{\bar{0}}(x, 2) = \frac{1}{\binom{N}{2}} \left(2x^2 - 2Nx + \binom{N}{2} \right)$$

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If you're checking out Wikipedia, they're equal up to a binom coeff.

An example, cont.

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Also: 3-term recurrence relation.

Multivariate Krawtchouk

Need $K := (p, \tilde{p}, U)$: (TLDR: more parameters; generalize p_0, p_1)

1. scalar tuples $p, \tilde{p} \in \mathbb{C}^{m+1}$
2. complex $(m+1) \times (m+1)$ matrix U

satisfying

1. $p_0 = \tilde{p}_0 \neq 0$;
2. $U = (u_{i,j})_{i,j=0}^m$ satisfies the normalization condition $u_{0,j} = u_{i,0} = 1$ for $i, j \in [m]$;
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Related to character algebras.

Krawtchouk polynomials

1. $\{t_i : i \in [m]\}$: $m + 1$ commuting indeterminates.
2. $D \in \mathbb{Z}_{\geq 0}$: a fixed degree parameter
3. $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_m), \tilde{\alpha} = (\tilde{\alpha}_0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_m) \in \mathbb{Z}_{\geq 0}^{m+1}$ such that $|\alpha| = |\tilde{\alpha}| = D$.

$$\prod_{i=0}^m \left(\sum_{j=0}^m u_{i,j} t_j \right)^{\tilde{\alpha}_i} = \sum_{\alpha} \frac{D!}{\alpha!} \mathcal{P}_{\tilde{0}}(\alpha, \tilde{\alpha}) t^{\alpha}. \quad (1)$$

(so in $U = \begin{pmatrix} 1 & 1 \\ 1 & -p_0/p_1 \end{pmatrix}$ in the previous example with $t_0 := 1$)

$$(1+t)^{N-x} \left(1 - \frac{p_0}{p_1} t\right)^x = \sum_{m=0}^N \binom{N}{m} \mathcal{P}_{\tilde{0}}(x, m; N, p_1) t^m.$$

Odd Krawtchouk polynomials

Let $\Lambda := (q, \tilde{q}, V)$ be another tuple satisfying similar relations

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$$\prod_{i=0}^n \left(\sum_{j=0}^n v_{i,j} \psi_j \right)^{\tilde{\epsilon}_i} = \sum_{\epsilon} D! \mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon}) \psi^{\epsilon}, \quad (2)$$

where the products are taken in the normal order $\psi_0^{\epsilon_0} \cdots \psi_n^{\epsilon_n}$.

Odd Krawtchouk polynomials

Explicitly: $A(\epsilon, \tilde{\epsilon}) := \text{diag}(\tilde{\epsilon})V \text{diag}(\epsilon)$. Then

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Reversing the roles of the ϵ and $\tilde{\epsilon}$ amounts to transposing the matrix V .

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A uniform way to get super Krawtchouk polynomials: put K and Λ together.

Write

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Introduce $\{z_k : k \in \mathbb{I}_{m|n}\}$ with $z_i := t_i$ (commutative variables), for $i \in [m]$ and $z_{m+1+j} := \psi_j$ (anticommutative) for $j \in [n]$. t_i and ψ_j commute.

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Set $(A|B) := \text{diag}(A|B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Set $Y := (y_{i,j})_{i,j \in \mathbb{I}_{m|n}} = (U|V)$. Then

$$(p_0^{-1}P|q_0^{-1}Q)Y(\tilde{P}|\tilde{Q})Y^t = I_{m+n+2}.$$

Generating Function

Let $\mathbb{A}_{m|n} := \mathbb{Z}_{\geq 0}^{m+1} \times \mathbb{Z}_2^{n+1}$, $\mathbb{A}_{m|n}^D := \{(\alpha, \epsilon) \in \mathbb{A}_{m|n} : |(\alpha, \epsilon)| = D\}$.

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For $(\tilde{\alpha}, \tilde{\epsilon}) \in \mathbb{A}_{m|n}^D$ we write

$$\prod_{i \in \mathbb{I}_{m|n}} \left(\sum_{j \in \mathbb{I}_{m|n}} y_{i,j} z_j \right)^{(\tilde{\alpha}, \tilde{\epsilon})_i} = \sum_{(\alpha, \epsilon) \in \mathbb{A}_{m|n}^D} \frac{D!}{\alpha!} \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}; \mathbf{K}, \Lambda, D) z^{(\alpha, \epsilon)}. \quad (4)$$

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Thus

$$\mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) = \binom{D}{d}^{-1} \mathcal{P}_{\bar{0}}(\alpha, \tilde{\alpha}; \mathbf{K}, D-d) \mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon}; \Lambda, d). \quad (5)$$

Note $\mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) = 0$ unless $(\alpha, \epsilon), (\tilde{\alpha}, \tilde{\epsilon}) \in \mathbb{A}_{m|n}^{D-d,d} := \{(\alpha, \epsilon) \in \mathbb{A}_{m|n}^D : |\epsilon| = d\}$.

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$$[X, Y] = XY - (-1)^{|X||Y|} YX.$$

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Data needed soon: 2 Cartans \mathfrak{h} : standard diagonal; $\tilde{\mathfrak{h}}$: non-diagonal, obtained via certain change of basis using $U, V, P, Q, \tilde{P}, \tilde{Q}$ from K, Λ .

An antiautomorphism φ : also defined using K, Λ , and the supertranspose.
 φ fixed both $\mathfrak{h}, \tilde{\mathfrak{h}}$.

Superpolynomial module

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Example. $\partial_{\xi_2} \xi_1 \xi_2 = 0 + (-1) \xi_1 \partial_{\xi_2} \xi_2 = -\xi_1$

Write

$$w_i := x_i, \text{ for } i \in [m]; \quad w_{m+1+j} := \xi_j \text{ for } j \in [n]; \quad \partial_i = \partial_{w_i} \text{ for } i \in \mathbb{I}_{m|n}.$$

Then $E_{i,j} \in \mathfrak{g}$ acts by

$$\rho : E_{i,j} \mapsto w_i \partial_j$$

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θ, κ : Explicit const produced by K, Λ resp.

Theorem A

The super Krawtchouk polynomials $\mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) = \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}; K, \Lambda, D)$ arise, up to explicit scalars, as entries of the transition matrices between the \mathfrak{h} - and $\tilde{\mathfrak{h}}$ -weight bases for the module \mathfrak{P}^D . Specifically, we have

$$\tilde{x}^{\tilde{\alpha}} \tilde{\xi}^{\tilde{\epsilon}} = \tilde{\theta}^{D-d} \tilde{\kappa}^d D! \sum_{(\alpha, \epsilon) \in \mathbb{A}_{m|n}^{D-d, d}} \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) \frac{\tilde{p}^\alpha \tilde{q}^\epsilon}{\alpha!} x^\alpha \xi^\epsilon, \quad (6)$$

$$x^\alpha \xi^\epsilon = \theta^{D-d} \kappa^d D! \sum_{(\tilde{\alpha}, \tilde{\epsilon}) \in \mathbb{A}_{m|n}^{D-d, d}} \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) \frac{p^{\tilde{\alpha}} q^{\tilde{\epsilon}}}{\tilde{\alpha}!} \tilde{x}^{\tilde{\alpha}} \tilde{\xi}^{\tilde{\epsilon}}. \quad (7)$$

Orthogonality

Theorem B

Let $(\alpha, \epsilon), (\tilde{\alpha}, \tilde{\epsilon}), (\beta, \eta), (\tilde{\beta}, \tilde{\eta}) \in \mathbb{A}_{m|n}^{D-d,d}$. The super Krawtchouk polynomials satisfy the following orthogonality condition

$$\sum_{(\alpha, \epsilon) \in \mathbb{A}_{m|n}^{D-d,d}} \tilde{p}^\alpha \tilde{q}^\epsilon \frac{D!}{\alpha!} \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) \mathcal{P}(\alpha, \epsilon, \tilde{\beta}, \tilde{\eta}) = \delta_{\tilde{\alpha}, \tilde{\beta}} \delta_{\tilde{\epsilon}, \tilde{\eta}} \frac{\tilde{\alpha}!}{D!} \frac{p_0^{D-d} q_0^d}{\tilde{p}^{\tilde{\alpha}} \tilde{q}^{\tilde{\epsilon}}},$$

$$\sum_{(\tilde{\alpha}, \tilde{\epsilon}) \in \mathbb{A}_{m|n}^{D-d,d}} p^{\tilde{\alpha}} q^{\tilde{\epsilon}} \frac{D!}{\tilde{\alpha}!} \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) \mathcal{P}(\beta, \eta, \tilde{\alpha}, \tilde{\epsilon}) = \delta_{\alpha, \beta} \delta_{\epsilon, \eta} \frac{\alpha!}{D!} \frac{p_0^{D-d} q_0^d}{\tilde{p}^\alpha \tilde{q}^\epsilon}.$$

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- On \mathfrak{P}^D , we set

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- A key point: $\langle -, - \rangle$ is still diagonal with respect to the tilde basis $\{\tilde{x}^\alpha \tilde{\xi}^\eta : (\alpha, \epsilon) \in \mathbb{A}_{m|n}^D\}$

$$\langle \tilde{x}^\alpha \tilde{\xi}^\epsilon, \tilde{x}^\beta \tilde{\xi}^\eta \rangle = \delta_{\alpha,\beta} \delta_{\epsilon,\eta} \frac{\alpha!}{p^\alpha q^\epsilon} \tilde{\theta}^{|\alpha|} \tilde{\kappa}^{|\epsilon|}, \text{ for all } (\alpha, \epsilon), (\beta, \eta) \in \mathbb{A}_{m|n}^D.$$

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Definition (Multi-color super grading)

On $\mathbb{I}_{m|n} = \{0, \dots, m | m+1, \dots, m+n+1\}$ set $|i| = 0$ if $i \leq m$ and $|i| = 1$ if $i \geq m+1$. Let v_k denote the k -th standard basis of \mathbb{Z}_2^{n+1} . Whenever $k < 0$, the symbol v_k is regarded as 0.

For $E_{i,j} \in \mathfrak{g}$, we set

$$\overline{E_{i,j}} := |i|v_{i-(m+1)} + |j|v_{j-(m+1)} \in \mathbb{Z}_2^{n+1}.$$

Equivalently, \mathfrak{g} acquires a grading from its nat. rep $\mathfrak{P}^1 = \text{Span}\{x_i, \xi_j\}$ where ξ_j has deg $v_j \in \mathbb{Z}_2^{n+1}$ and x_i 's has deg 0.

Orthogonality

Explicitly, for $i, j \in \mathbb{I}_{m|n}$

$$\overline{E_{i,j}} = \begin{cases} 0, & |i| = |j| = 0 \\ v_{i-(m+1)} + v_{j-(m+1)}, & |i| = |j| = 1 \\ v_{i-(m+1)}, & |i| = 1, |j| = 0 \\ v_{j-(m+1)} & |i| = 0, |j| = 1 \end{cases}.$$

Let $\mathfrak{g}_\epsilon := \text{Span}\{E_{i,j} : \overline{E_{i,j}} = \epsilon\}$ for $\epsilon \in \mathbb{Z}_2^{n+1}$. Thus

$$\mathfrak{g} = \bigoplus_{\epsilon \in \mathbb{Z}_2^{n+1}} \mathfrak{g}_\epsilon.$$

From a physics point of view, it records which fermionic positions in $E_{i,j}$ modulo 2.

Orthogonality

The usual \mathbb{Z}_2 -super grading is consistent with this multi-color \mathbb{Z}_2^{n+1} -grading:

$$[\mathfrak{g}_\epsilon, \mathfrak{g}_\eta] \subseteq \mathfrak{g}_{\epsilon+\eta}.$$

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In the sense of [Sch79], this gives a \mathbb{Z}_2^{n+1} -graded π -Lie algebra structure where $\pi : \mathbb{Z}_2^{n+1} \rightarrow \mathbb{C}^\times : \eta \mapsto (-1)^{|\eta|}$ is a character; compare [Pri97].

Orthogonality

The \mathfrak{g} -mod \mathfrak{P}^D has a compatible grading too.

For $x^\alpha \xi^\epsilon \in \mathfrak{P}^D$:

$$\overline{x^\alpha \xi^\epsilon} := (\epsilon_0, \dots, \epsilon_n) \in \mathbb{Z}_2^{n+1}, \quad \text{then} \quad \overline{E_{i,j} \cdot x^\alpha \xi^\epsilon} = \overline{E_{i,j}} + \overline{x^\alpha \xi^\epsilon}.$$

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Proposition

For all homogeneous $X \in \mathfrak{g}$ and $u, v \in \mathfrak{P}^D$, we have

$$\langle X.u, v \rangle = (-1)^{|X| |\overline{X} \cdot \bar{u}|} \langle u, \varphi(X).v \rangle. \quad (8)$$

(\cdot means dot prod. on \mathbb{Z}_2^{n+1})

Here φ is a map that “captures” K, Λ and the supertranspose in \mathfrak{g} .

Fock Space Set-up

Goal: odd Krawtchouk \rightsquigarrow Fermionic prob (story 1) & spherical function (story 2)

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Antisymmetry of the exterior products = Pauli exclusion principle ($e_i \wedge e_i = 0$)

Oriented Grassmannian

- For any oriented orthonormal basis $\{f_0, f_1, \dots, f_{d-1}\}$ for \mathcal{W} , $f_0 \wedge f_1 \wedge \dots \wedge f_{d-1}$ uniquely corresponds to an oriented d -dimensional subspace of \mathcal{W} spanned by f_i 's.

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- Let $G := SO(n+1)$ and $K := SO(d) \times SO(n+1-d)$.

$$\widetilde{Gr}(n+1, d) := SO(n+1)/SO(d) \times SO(n+1-d) = G/K.$$

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$$\text{Fock} \rightsquigarrow G/K$$

Story 1: Fermion Probabilities

- For $I = \{i_k : i_0 < i_1 < \dots < i_{d-1}\} \subseteq [n]$ set $e_I := e_{i_0} \wedge e_{i_1} \wedge \dots \wedge e_{i_{d-1}}$. Then for any $g \in GL$

$$g.e_J = \sum_{|I|=d} \det(g_{I,J}) e_I. \quad (9)$$

(coefficients are the Plücker coordinates of $g.P_J = \text{Span}\{e_j : j \in J\}$)

- Fix $|J| = d$. For $g \in SO(n+1)$ the vector $g.e_J$ as in Eq. (9) has norm 1. Hence

$$\mathbb{P}_{I|J} := |\det(g_{I,J})|^2 \quad (10)$$

form a probability distribution on the set of size- d index subsets $\{I \subseteq [n] : |I| = d\}$.

- In the fermionic interpretation of $\wedge^d \mathcal{W}$, this is the probability distribution obtained by expressing the state $g.e_J$ in the occupation basis $\{e_I : |I| = d\}$, cf. [Got07, Section 2].

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Fock \rightsquigarrow Ferm. prob.

Story 2: Rep theory

- $\mathcal{W} = \mathbb{R}^{n+1}$: natural $G = SO(n+1)$ -mod. G acts on \mathcal{F} via

$$g.(u_0 \wedge \cdots \wedge u_{d-1}) := (g.u_0) \wedge \cdots \wedge (g.u_{d-1}).$$

- $(G, K) = (SO(n+1), SO(d) \times SO(n+1-d))$: G -mod $\bigwedge^d \mathcal{W}$ is typically irreducible, except a middle case when $n+1$ is even and $d = \frac{n+1}{2}$. Here $\bigwedge^d \mathcal{W}$ splits into two irreducible components.
- These mods are K -spherical

$$v^K := e_0 \wedge e_1 \wedge \cdots \wedge e_{d-1} \in \bigwedge^d \mathcal{W}.$$

Since for $(k_0, k_1) \in SO(d) \times SO(n+1-d)$,

$$(k_0, k_1).v^K = k_0(e_0 \wedge \cdots \wedge e_{d-1}) = (\det k_0)v^K = v^K.$$

Zonal spherical function

- The associated (zonal) spherical function is

$$\phi_d(g) := (v^K, g.v^K), \quad g \in G.$$

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- Recall $QV\tilde{Q}V^t = q_0 I_{n+1}$ for $\Lambda = (q, \tilde{q}, V)$. May pick $Q^{1/2}$ and $\tilde{Q}^{1/2}$ so that

$$g = g_V := q_0^{-1/2} Q^{1/2} V \tilde{Q}^{1/2} \in SO(n+1). \quad (11)$$

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- G acts transitively on oriented bases, may choose $\sigma_I, \sigma_J \in G$ such that

$$\sigma_I.v^K = e_I, \quad \sigma_J.v^K = e_J.$$



$$\begin{aligned}
 (e_I, g.e_J) &= (\sigma_I v^K, g\sigma_J.v^K) \\
 &= (v^K, \sigma_I^{-1}g\sigma_J.v^K) \\
 &= \phi_d(\sigma_I^{-1}g\sigma_J).
 \end{aligned} \tag{12}$$

Choice-independent by the bi- K -invariance!

- Combing everything all together,

$$\mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon}) = \frac{q_0^{d/2}}{d!} \sum_{|I|=|J|=d} q_I^{-\frac{1}{2}} \tilde{q}_J^{-\frac{1}{2}} \phi_d(\sigma_I^{-1}g_V\sigma_J)\tilde{\epsilon}_I\epsilon_J. \tag{13}$$

Coefficients of $\mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon}) =$ weighted evaluation of ϕ_d of $\widetilde{Gr}(n+1, d)$.

Fock $\rightsquigarrow G/K \rightsquigarrow$ Sph. func. evaluations







Interpretation

•

$$\mathcal{P}_1(\epsilon, \tilde{\epsilon}) = \frac{q_0^{d/2}}{d!} \sum_{|I|=|J|=d} q_I^{-\frac{1}{2}} \tilde{q}_J^{-\frac{1}{2}} \phi_d(\sigma_I^{-1} g_V \sigma_J) \tilde{\epsilon}_I \epsilon_J. \quad (14)$$

In view of the probability interpretation (story 1) above (Eq. (10)), the squared norms of the evaluation of ϕ_d are the transition probabilities of different occupation basis dictated by $\Lambda = (q, \tilde{q}, V)$.

Fock \rightsquigarrow G/K \rightsquigarrow Sph. func. evaluations \sim Fermionic Prob.

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