Super Krawtchouk Polynomials via Lie Superalgebras

Joint work with Plamen Iliev

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- They are put in the framework of Lie superalgebras ($\mathfrak{gl}$). *Something special.*

- They are orthogonal. *Using rep theory.*

- They have (fermionic) probability interpretation. *"Quantum" stuff.*

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- Griffiths also proposed a multivariate version
- Iliev gave a Lie theoretic interpretation [Ili12]

An example

Fix $N \in \mathbb{Z}_{\geq 0}$. For $m, x \in \{0, 1, \ldots, N\} =: [N]$ set

$$(1+t)^{N-x}(1-t)^{x} = \sum_{m=0}^{N} \binom{N}{m} \mathcal{P}_{\overline{0}}(x, m; \underbrace{N, 1/2}_{\text{optional}})\, t^{m}.$$

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Explicitly

$$\mathcal{P}_{\overline{0}}(x, 0) = 1, \quad \mathcal{P}_{\overline{0}}(x, 1) = \frac{1}{N}(-2x+N), \quad \mathcal{P}_{\overline{0}}(x, 2) = \frac{1}{\binom{N}{2}} \left(2x^{2} - 2Nx + \binom{N}{2} \right)$$

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Wikipedia formulas equal up to a binom coeff.

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Also: 3-term recurrence relation.

Multivariate Krawtchouk

Need K := $(p, \tilde{p}, U)$: (TLDR: more parameters; generalize p_0, p_1)

1. scalar $m+1$-tuples $p, \tilde{p} \in \mathbb{C}^{m+1}$;
2. complex $(m+1) \times (m+1)$ matrix U,

satisfying

1. $p_0 = \tilde{p}_0 \neq 0$;
2. $U = (u_{i,j})_{i,j=0}^{m}$ satisfies the normalization condition $u_{0,j} = u_{i,0} = 1$ for $i, j \in [m]$;
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Related to character algebras.

Multivariate Krawtchouk

1. $\{t_i : i \in [m]\}$: $m+1$ commuting indeterminates.
2. $D \in \mathbb{Z}_{\geq 0}$: **total deg.**
3. **deg. indices** $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m), \tilde{\alpha} = (\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) \in \mathbb{Z}^{m+1}_{\geq 0}$ such that $|\alpha| = |\tilde{\alpha}| = D$.

$$\prod_{i=0}^{m} \left(\sum_{j=0}^{m} u_{i,j} t_j \right)^{\tilde{\alpha}_i} = \sum_{\alpha} \frac{D!}{\alpha!} \mathcal{P}_{\overline{0}}(\alpha, \tilde{\alpha}) t^{\alpha}. \tag{1}$$

(Setting $U = \begin{pmatrix} 1 & 1 \\ 1 & -p_0/p_1 \end{pmatrix}$ and $t_0 := 1$ & $t_1 = t$ gives the 1-var example:

$$(1+t)^{N-x} \left(1 - \frac{p_0}{p_1} t\right)^{x} = \sum_{m=0}^{N} \binom{N}{m} \mathcal{P}_{\overline{0}}(x, m; N, p_1)\, t^m.)$$

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Let $\{\psi_j : j \in [n]\}$ be the Grassmann variables:

$$\psi_i\psi_j = -\psi_j\psi_i, \quad \text{for all } i, j \in [n], \quad (\Rightarrow {\psi_j}^2 = 0)$$

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$$\prod_{i=0}^{n} \left(\sum_{j=0}^{n} v_{i,j}\psi_j \right)^{\tilde{\epsilon}_i} = \sum_{\epsilon} D! \mathcal{P}_{\overline{1}}(\epsilon, \tilde{\epsilon})\psi^{\epsilon}, \tag{2}$$

where $\psi^{\epsilon} = \psi_0^{\epsilon_0} \cdots \psi_n^{\epsilon_n}$ are in the ascending order.

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Explicitly: $A(\epsilon, \tilde{\epsilon}) := \operatorname{diag}(\tilde{\epsilon}) V \operatorname{diag}(\epsilon)$. Then

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Reversing the roles of the ϵ and $\tilde{\epsilon}$ amounts to transposing the matrix V.

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To get super Krawtchouk polynomials: put K and Λ together.

Write

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Put t_i and ξ_j together, write down a similar gen. func., collet the terms, and you get

$$\mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) = \binom{D}{d}^{-1} \mathcal{P}_{\overline{0}}(\alpha, \tilde{\alpha}; \mathrm{K}, D-d)\mathcal{P}_{\overline{1}}(\epsilon, \tilde{\epsilon}; \Lambda, d). \tag{4}$$

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Here $D = |(\alpha, \epsilon)|$ and $d = |\epsilon|$.
Stuff I'm hiding under the rug Set $(A|B) := \mathrm{diag}(A|B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Set $Y := (y_{i,j})_{i,j \in \mathbb{I}_{m|n}} = (U|V)$. Then

$$(p_0^{-1} P | q_0^{-1} Q) Y (\tilde{P}|\tilde{Q}) Y^{\mathrm{t}} = I_{m+n+2}.$$

Generating Function

Let $\mathbb{A}_{m|n} := \mathbb{Z}_{\geq 0}^{m+1} \times \mathbb{Z}_2^{n+1}, \quad \mathbb{A}_{m|n}^D := \{(\alpha, \epsilon) \in \mathbb{A}_{m|n} : |(\alpha, \epsilon)| = D\}$.

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$$\prod_{i \in \mathbb{I}_{m|n}} \left(\sum_{j \in \mathbb{I}_{m|n}} y_{i,j} z_j \right)^{(\tilde{\alpha},\tilde{\epsilon})_i} = \sum_{(\alpha,\epsilon) \in \mathbb{A}_{m|n}^D} \frac{D!}{\alpha!} \mathcal{P}(\alpha,\epsilon,\tilde{\alpha},\tilde{\epsilon}; \mathrm{K}, \Lambda, D) z^{(\alpha,\epsilon)}. \tag{5}$$

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Thus

$$\mathcal{P}(\alpha,\epsilon,\tilde{\alpha},\tilde{\epsilon}) = \binom{D}{d}^{-1} \mathcal{P}_{\overline{0}}(\alpha,\tilde{\alpha};\mathrm{K},D-d)\mathcal{P}_{\overline{1}}(\epsilon,\tilde{\epsilon};\Lambda,d). \tag{6}$$

Note $\mathcal{P}(\alpha,\epsilon,\tilde{\alpha},\tilde{\epsilon}) = 0$ unless
$(\alpha,\epsilon),(\tilde{\alpha},\tilde{\epsilon}) \in \mathbb{A}_{m|n}^{D-d,d} := \{(\alpha,\epsilon) \in \mathbb{A}_{m|n}^D : |\epsilon| = d\}$.

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$$[X,Y] = XY - (-1)^{|X||Y|} YX.$$

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$\mathfrak{g}$ has:

• 2 Cartans $\mathfrak{h}$ = standard diagonal; $\tilde{\mathfrak{h}}$ = some non-diagonal, obtained via certain change of basis using $U, V, P, Q, \tilde{P}, \tilde{Q}$ from K, Λ.

• An antiautomorphism φ: also defined using K, Λ, and the supertranspose. φ fixed both $\mathfrak{h}, \tilde{\mathfrak{h}}$.

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$$w_i := x_i, \text{ for } i \in [m]; \quad w_{m+1+j} := \xi_j \text{ for } j \in [n]; \quad \partial_i = \partial_{w_i} \text{ for } i \in \mathbb{I}_{m|n}.$$

Then $E_{i,j} \in \mathfrak{g}$ acts on $\mathfrak{P}^D$ by

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Example. $\partial_{\xi_2}\xi_1\xi_2 = 0 + (-1)\xi_1\partial_{\xi_2}\xi_2 = -\xi_1$ *obeys the super Leibniz rule.*

As usual $x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_m^{\alpha_m} \xi_0^{\epsilon_0} \cdots \xi_n^{\epsilon_n} =: x^\alpha \xi^\epsilon$ form an $\mathfrak{h}$-weight basis

$$\{x^\alpha \xi^\epsilon : (\alpha, \epsilon) \in \mathbb{A}_{m|n}^D\}.$$

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θ, κ: Explicit const produced by K, Λ resp.

Theorem A

The super Krawtchouk polynomials $\mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) = \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}; \mathrm{K}, \Lambda, D)$ arise, up to explicit scalars, as entries of the transition matrices between the $\mathfrak{h}$- and $\tilde{\mathfrak{h}}$-weight bases for the module $\mathfrak{P}^D$. Specifically, we have

$$\tilde{x}^{\tilde{\alpha}} \tilde{\xi}^{\tilde{\epsilon}} = \tilde{\theta}^{D-d} \tilde{\kappa}^d D! \sum_{(\alpha,\epsilon) \in \mathbb{A}^{D-d,d}_{m|n}} \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) \frac{\tilde{p}^\alpha \tilde{q}^\epsilon}{\alpha!} x^\alpha \xi^\epsilon, \tag{7}$$

$$x^\alpha \xi^\epsilon = \theta^{D-d} \kappa^d D! \sum_{(\tilde{\alpha},\tilde{\epsilon}) \in \mathbb{A}^{D-d,d}_{m|n}} \mathcal{P}(\alpha, \epsilon, \tilde{\alpha}, \tilde{\epsilon}) \frac{p^{\tilde{\alpha}} q^{\tilde{\epsilon}}}{\tilde{\alpha}!} \tilde{x}^{\tilde{\alpha}} \tilde{\xi}^{\tilde{\epsilon}}. \tag{8}$$

Weight bases

$\{x^\alpha \xi^\epsilon : (\alpha, \epsilon) \in \mathbb{A}^D_{m|n}\}$: an $\mathfrak{h}$-weight basis.
$\{\tilde{x}^\beta \tilde{\xi}^\eta : (\alpha, \epsilon) \in \mathbb{A}^D_{m|n}\}$: an $\tilde{\mathfrak{h}}$-weight basis. (Introduced using data K, Λ)
θ, κ: Explicit const produced by K, Λ resp.

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Proof by computation.

Orthogonality

Theorem B

Let $(\alpha,\epsilon),(\tilde{\alpha},\tilde{\epsilon}),(\beta,\eta),(\tilde{\beta},\tilde{\eta})\in\mathbb{A}_{m|n}^{D-d,d}$. *The super Krawtchouk polynomials satisfy the following orthogonality condition*

$$\sum_{(\alpha,\epsilon)\in\mathbb{A}_{m|n}^{D-d,d}}\tilde{p}^{\alpha}\tilde{q}^{\epsilon}\frac{D!}{\alpha!}\mathcal{P}(\alpha,\epsilon,\tilde{\alpha},\tilde{\epsilon})\mathcal{P}(\alpha,\epsilon,\tilde{\beta},\tilde{\eta})=\delta_{\tilde{\alpha},\tilde{\beta}}\delta_{\tilde{\epsilon},\tilde{\eta}}\frac{\tilde{\alpha}!}{D!}\frac{p_0^{D-d}q_0^{d}}{p^{\tilde{\alpha}}q^{\tilde{\epsilon}}},$$

$$\sum_{(\tilde{\alpha},\tilde{\epsilon})\in\mathbb{A}_{m|n}^{D-d,d}}p^{\tilde{\alpha}}q^{\tilde{\epsilon}}\frac{D!}{\tilde{\alpha}!}\mathcal{P}(\alpha,\epsilon,\tilde{\alpha},\tilde{\epsilon})\mathcal{P}(\beta,\eta,\tilde{\alpha},\tilde{\epsilon})=\delta_{\alpha,\beta}\delta_{\epsilon,\eta}\frac{\alpha!}{D!}\frac{p_0^{D-d}q_0^{d}}{\tilde{p}^{\alpha}\tilde{q}^{\epsilon}}.$$

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$$\langle x^\alpha \xi^\epsilon, x^\beta \xi^\eta \rangle := \delta_{\alpha,\beta}\delta_{\epsilon,\eta}\frac{\alpha!}{\tilde{p}^\alpha \tilde{q}^\epsilon}\theta^{|\alpha|}\kappa^{|\epsilon|}, \text{ for all } (\alpha,\epsilon),(\beta,\eta) \in \mathbb{A}^D_{m|n}.$$

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- A key point: $\langle -, - \rangle$ is still diagonal with respect to the tilde basis $\{\tilde{x}^\beta \tilde{\xi}^\eta : (\alpha,\epsilon)\in \mathbb{A}^D_{m|n}\}$

$$\langle \tilde{x}^\alpha \tilde{\xi}^\epsilon, \tilde{x}^\beta \tilde{\xi}^\eta \rangle = \delta_{\alpha,\beta}\delta_{\epsilon,\eta}\frac{\alpha!}{p^\alpha q^\epsilon}\tilde{\theta}^{|\alpha|}\tilde{\kappa}^{|\epsilon|}, \text{ for all } (\alpha,\epsilon),(\beta,\eta)\in \mathbb{A}^D_{m|n}.$$

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Definition (Multi-color super grading)

On $\mathbb{I}_{m|n} = \{0, \ldots, m|m+1, \ldots, m+n+1\}$ *set* $|i| = 0$ *if* $i \leq m$ *and* $|i| = 1$ *if* $i \geq m+1$. *Let* v_k *denote the* k*-th standard basis of* $\mathbb{Z}_2^{n+1}$. *Whenever* $k < 0$, *the symbol* v_k *is regarded as 0.*
For $E_{i,j} \in \mathfrak{g}$, *define*

$$\overline{E_{i,j}} := |i|v_{i-(m+1)} + |j|v_{j-(m+1)} \in \mathbb{Z}_2^{n+1}.$$

E.g. In $\mathfrak{gl}(2|3)$ with $\mathbb{I}_{1|2} = \{0, 1|2, 3, 4\}$, $\overline{E_{0,2}} = (1,0,0)$, $\overline{E_{3,4}} = (0,1,1)$ while $\overline{E_{4,4}} = (0,0,0)$. *Records which odd indices occur mod 2.*

Orthogonality

Let $\mathfrak{g}_\epsilon := \operatorname{Span}\{E_{i,j} : \overline{E_{i,j}} = \epsilon\}$ for $\epsilon \in \mathbb{Z}_2^{n+1}$. Thus

$$\mathfrak{g} = \bigoplus_{\epsilon \in \mathbb{Z}_2^{n+1}} \mathfrak{g}_\epsilon.$$

From a physics point of view, it records which fermionic positions in $E_{i,j}$ modulo 2.

Orthogonality

The usual $\mathbb{Z}_2$-super grading is consistent with this multi-color $\mathbb{Z}_2^{n+1}$-grading:

$$[\mathfrak{g}_\epsilon, \mathfrak{g}_\eta] \subseteq \mathfrak{g}_{\epsilon+\eta}.$$

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In the sense of [Sch79], this gives a $\mathbb{Z}_2^{n+1}$-graded π-Lie algebra structure where $\pi : \mathbb{Z}_2^{n+1} \to \mathbb{C}^\times : \eta \mapsto (-1)^{|\eta|}$ is a character; compare [Pri97].

Orthogonality

The $\mathfrak{g}$-mod $\mathfrak{P}^D$ has a compatible grading too.

For $x^\alpha \xi^\epsilon \in \mathfrak{P}^D$:

$$\overline{x^\alpha \xi^\epsilon} := (\epsilon_0, \ldots, \epsilon_n) \in \mathbb{Z}_2^{n+1}, \quad \text{then} \quad \overline{E_{i,j}.x^\alpha \xi^\epsilon} = \overline{E_{i,j}} + \overline{x^\alpha \xi^\epsilon}.$$

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Proposition

For all homogeneous $X \in \mathfrak{g}$ and $u, v \in \mathfrak{P}^D$, we have

$$\langle X.u,\, v\rangle = (-1)^{|X|\overline{X}\cdot\overline{u}} \langle u, \varphi(X).v\rangle. \tag{9}$$

($\cdot$ means dot prod. on $\mathbb{Z}_2^{n+1}$)

Here φ is a map that "captures" K, Λ and the supertranspose in $\mathfrak{g}$.

Fock Space Set-up

Goal: odd Krawtchouk $\rightsquigarrow$ Fermionic prob (story 1) & spherical function (story 2)

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Antisymmetry of the exterior products = Pauli exclusion principle ($e_i \wedge e_i = 0$)

Oriented Grassmannian

- For any oriented orthonormal basis $\{f_0, f_1, \ldots, f_{d-1}\}$ for $\mathcal{W}$, $f_0 \wedge f_1 \wedge \cdots \wedge f_{d-1}$ uniquely corresponds to an oriented d-dimensional subspace of $\mathcal{W}$ spanned by f_i's.

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- Let $G := SO(n+1)$ and $K := SO(d) \times SO(n+1-d)$.

$$\widetilde{Gr}(n+1, d) := SO(n+1)/SO(d) \times SO(n+1-d) = G/K.$$

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$$\text{Fock} \rightsquigarrow G/K$$

Story 1: Fermion Probabilities

• For $I = \{i_k : i_0 < i_1 < \cdots < i_{d-1}\} \subseteq [n]$ set $e_I := e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge e_{i_{d-1}}$. Then for any $g \in GL$

$$g.e_J = \sum_{|I|=d} \det(g_{I,J}) e_I. \tag{10}$$

(coefficients are the Plücker coordinates of $g.P_J = \mathrm{Span}\{e_j : j \in J\}$)
• Fix $|J| = d$. For $g \in SO(n+1)$ the vector $g.e_J$ as in Eq. (10) has norm 1. Hence

$$\mathbb{P}_{I|J} := \left|\det(g_{I,J})\right|^2 \tag{11}$$

form a probability distribution on the set of size-d index subsets $\{I \subseteq [n] : |I| = d\}$.
• In the fermionic interpretation of $\wedge^d \mathcal{W}$, this is the probability distribution obtained by expressing the state $g.e_J$ in the occupation basis $\{e_I : |I| = d\}$, cf. [Got07, Section 2].

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Fock $\rightsquigarrow$ Ferm. prob.

Story 2: Rep theory

- $\mathcal{W} = \mathbb{R}^{n+1}$: natural $G = SO(n+1)$-mod. G acts on $\mathcal{F}$ via

$$g.(u_0 \wedge \cdots \wedge u_{d-1}) := (g.u_0) \wedge \cdots \wedge (g.u_{d-1}).$$

- $(G, K) = (SO(n+1), SO(d) \times SO(n+1-d))$: G-mod $\bigwedge^d \mathcal{W}$ is typically irreducible, except a middle case when $n+1$ is even and $d = \dfrac{n+1}{2}$. Here $\bigwedge^d \mathcal{W}$ splits into two irreducible components.
- These mods are K-spherical

$$v^K := e_0 \wedge e_1 \wedge \cdots \wedge e_{d-1} \in \bigwedge^d \mathcal{W}.$$

Since for $(k_0, k_1) \in SO(d) \times SO(n+1-d)$,

$$(k_0, k_1).v^K = k_0(e_0 \wedge \cdots \wedge e_{d-1}) = (\det k_0) v^K = v^K.$$

Zonal spherical function

- The associated (zonal) spherical function is

$$\phi_d(g) := (v^K,\ g.v^K), \qquad g \in G.$$

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- Recall $QV\tilde{Q}V^{\mathrm{t}} = q_0 I_{n+1}$ for $\Lambda = (q, \tilde{q}, V)$. May pick $Q^{1/2}$ and $\tilde{Q}^{1/2}$ so that

$$g = g_V := q_0^{-1/2} Q^{1/2} V \tilde{Q}^{1/2} \in SO(n+1). \tag{12}$$

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det $g_{I,J}$ is exactly given by $(e_I, g.e_J)$.

- G acts transitively on oriented bases, may choose $\sigma_I, \sigma_J \in G$ such that

$$\sigma_I.v^K = e_I, \quad \sigma_J.v^K = e_J.$$

•

$$\begin{aligned}(e_I, g.e_J) &= (\sigma_I v^K, g\sigma_J.v^K)\\ &= (v^K, \sigma_I^{-1} g\sigma_J.v^K)\\ &= \phi_d(\sigma_I^{-1} g\sigma_J).\end{aligned} \tag{13}$$

Choice-independent by the bi-K-invariance!

• Combining everything all together,

$$\mathcal{P}_{\overline{1}}(\epsilon, \tilde{\epsilon}) = \frac{q_0^{d/2}}{d!} \sum_{|I|=|J|=d} q_I^{-\frac{1}{2}} \tilde{q}_J^{-\frac{1}{2}} \phi_d(\sigma_I^{-1} g_V \sigma_J)\tilde{\epsilon}_I \epsilon_J. \tag{14}$$

Coefficients of $\mathcal{P}_{\overline{1}}(\epsilon, \tilde{\epsilon})$ = weighted evaluation of ϕ_d of $\widetilde{Gr}(n+1, d)$.

$$\text{Fock} \rightsquigarrow G/K \rightsquigarrow \text{Sph. func. evaluations}$$

Interpretation

- $$\mathcal{P}_{\bar{1}}(\epsilon, \tilde{\epsilon}) = \frac{q_0^{d/2}}{d!} \sum_{|I|=|J|=d} q_I^{-\frac{1}{2}} \tilde{q}_J^{-\frac{1}{2}} \phi_d(\sigma_I^{-1} g_V \sigma_J) \tilde{\epsilon}_I \epsilon_J. \tag{15}$$

In view of the probability interpretation (story 1) above (Eq. (11)), the squared norms of the evaluation of ϕ_d are the transition probabilities of different occupation basis dictated by $\Lambda = (q, \tilde{q}, V)$.

$$\text{Fock} \rightsquigarrow G/K \rightsquigarrow \text{Sph. func. evaluations} \sim \text{Fermionic Prob.}$$

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