

From Shimura operators to the integrability of Calogero–Moser–Sutherland operators

Joint work with Siddhartha Sahi; with Plamen Iliev

Songhao Zhu

Georgia Institute of Technology
zhu.math@gatech.edu

ALGEBRA SEMINAR, UNIVERSITY OF GEORGIA
Oct. 12, 2026

Structure

Two main stories.

- Shimura operators
 - Background & Definition
 - Commutativity
 - A connection to symmetric functions

Background

1. Shimura: multivariate generalization of nearly holomorphic automorphic forms.

Background

1. Shimura: multivariate generalization of nearly holomorphic automorphic forms.
2. Introduced certain G -invariant differential operators on a Hermitian $X := G/K$
[Shi90] *Invariant differential operators on hermitian symmetric spaces, Ann. of Math.*

Background

1. Shimura: multivariate generalization of nearly holomorphic automorphic forms.
2. Introduced certain G -invariant differential operators on a Hermitian $X := G/K$
[Shi90] *Invariant differential operators on hermitian symmetric spaces*, *Ann. of Math.*
3. Wanted symmetric & non-negative ones spanning the space of inv. diff. op. $\mathfrak{D}(X)$

Background

1. Shimura: multivariate generalization of nearly holomorphic automorphic forms.
2. Introduced certain G -invariant differential operators on a Hermitian $X := G/K$
[Shi90] *Invariant differential operators on hermitian symmetric spaces*, *Ann. of Math.*
3. Wanted symmetric & non-negative ones spanning the space of inv. diff. op. $\mathfrak{D}(X)$
 - Simplest example: Laplace–Beltrami operator generalize Laplace operator

Background

1. Shimura: multivariate generalization of nearly holomorphic automorphic forms.
2. Introduced certain G -invariant differential operators on a Hermitian $X := G/K$
[Shi90] *Invariant differential operators on hermitian symmetric spaces, Ann. of Math.*
3. Wanted symmetric & non-negative ones spanning the space of inv. diff. op. $\mathfrak{D}(X)$
 - Simplest example: Laplace–Beltrami operator generalize Laplace operator
4. Sahi & Zhang’s [SZ19]: spectrum of *Shimura operators* $\sim$ Okounkov polynomials.

Background

1. Shimura: multivariate generalization of nearly holomorphic automorphic forms.
2. Introduced certain G -invariant differential operators on a Hermitian $X := G/K$
[Shi90] *Invariant differential operators on hermitian symmetric spaces, Ann. of Math.*
3. Wanted symmetric & non-negative ones spanning the space of inv. diff. op. $\mathfrak{D}(X)$
 - Simplest example: Laplace–Beltrami operator generalize Laplace operator
4. Sahi & Zhang’s [SZ19]: spectrum of *Shimura operators* $\sim$ Okounkov polynomials.
5. These are the even & symmetric polynomials with prescribed zeros (thus the word *interpolation*).

Background

1. Shimura: multivariate generalization of nearly holomorphic automorphic forms.
2. Introduced certain G -invariant differential operators on a Hermitian $X := G/K$
[Shi90] *Invariant differential operators on hermitian symmetric spaces, Ann. of Math.*
3. Wanted symmetric & non-negative ones spanning the space of inv. diff. op. $\mathfrak{D}(X)$
 - Simplest example: Laplace–Beltrami operator generalize Laplace operator
4. Sahi & Zhang’s [SZ19]: spectrum of *Shimura operators* $\sim$ Okounkov polynomials.
5. These are the even & symmetric polynomials with prescribed zeros (thus the word *interpolation*).
6. The theory of symmetric functions gives answers to Shimura’s problem.

In context (frakturs)

Let $X := G/K$ be as above. Change the point of view to (complexified) Lie algebras.
 $(\mathfrak{g}, \mathfrak{k})$: Harish-Chandra decomposition $[(-1, 0, 1)\text{-grading}]$

$$\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+ (= \mathfrak{k} \oplus \mathfrak{p})$$

$$[\mathfrak{k}, \mathfrak{p}^\pm] = \mathfrak{p}^\pm, \quad [\mathfrak{p}^+, \mathfrak{p}^+] = [\mathfrak{p}^-, \mathfrak{p}^-] = 0, \quad [\mathfrak{p}^+, \mathfrak{p}^-] \subseteq \mathfrak{k}$$

E.g. $(\mathfrak{gl}(2n), \mathfrak{gl}(n) \oplus \mathfrak{gl}(n))$: $\begin{pmatrix} \mathfrak{k} & \mathfrak{p}^+ \\ \mathfrak{p}^- & \mathfrak{k} \end{pmatrix}$

In context (fraktur)

1. $\mathfrak{D} := \mathfrak{D}(X)$: algebra of invariant differential operators on X .
 $\mathfrak{U} := \mathfrak{U}(\mathfrak{g})$; $\mathfrak{U}^{\mathfrak{k}} := \mathfrak{Z}_{\mathfrak{k}}(\mathfrak{U})$; $(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}} := \mathfrak{U}\mathfrak{k} \cap \mathfrak{U}^{\mathfrak{k}}$. Will also need the center $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{U})$.

In context (fracturs)

1. $\mathfrak{D} := \mathfrak{D}(X)$: algebra of invariant differential operators on X .
 $\mathfrak{U} := \mathfrak{U}(\mathfrak{g})$; $\mathfrak{U}^{\mathfrak{k}} := \mathfrak{Z}_{\mathfrak{k}}(\mathfrak{U})$; $(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}} := \mathfrak{U}\mathfrak{k} \cap \mathfrak{U}^{\mathfrak{k}}$. Will also need the center $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{U})$.
2. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$: Iwasawa decomposition (Group: $G = KAN$). Here $\mathfrak{a}$ is the Cartan subspace in $\mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{p}^+$. May identify $\mathfrak{S}(\mathfrak{a})$ with $\mathfrak{P}(\mathfrak{a}^*)$.

In context (fraġturs)

1. $\mathfrak{D} := \mathfrak{D}(X)$: algebra of invariant differential operators on X .
 $\mathfrak{U} := \mathfrak{U}(\mathfrak{g})$; $\mathfrak{U}^{\mathfrak{k}} := \mathfrak{Z}_{\mathfrak{k}}(\mathfrak{U})$; $(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}} := \mathfrak{U}\mathfrak{k} \cap \mathfrak{U}^{\mathfrak{k}}$. Will also need the center $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{U})$.
2. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$: Iwasawa decomposition (Group: $G = KAN$). Here $\mathfrak{a}$ is the Cartan subspace in $\mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{p}^+$. May identify $\mathfrak{S}(\mathfrak{a})$ with $\mathfrak{P}(\mathfrak{a}^*)$.
3. $\mathfrak{D} \cong \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$. This is the algebraic description of $\mathfrak{D}$.

In context (fracturs)

1. $\mathfrak{D} := \mathfrak{D}(X)$: algebra of invariant differential operators on X .
 $\mathfrak{U} := \mathfrak{U}(\mathfrak{g})$; $\mathfrak{U}^{\mathfrak{k}} := \mathfrak{Z}_{\mathfrak{k}}(\mathfrak{U})$; $(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}} := \mathfrak{U}\mathfrak{k} \cap \mathfrak{U}^{\mathfrak{k}}$. Will also need the center $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{U})$.
2. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$: Iwasawa decomposition (Group: $G = KAN$). Here $\mathfrak{a}$ is the Cartan subspace in $\mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{p}^+$. May identify $\mathfrak{S}(\mathfrak{a})$ with $\mathfrak{P}(\mathfrak{a}^*)$.
3. $\mathfrak{D} \cong \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$. This is the algebraic description of $\mathfrak{D}$.
4. $\gamma^0 : \mathfrak{D} \rightarrow \Lambda \subseteq \mathfrak{P}(\mathfrak{a}^*)$: the Harish-Chandra isomorphism into certain poly. ring.
 Good map! Essentially a symmetry-preserving projection.

In context (fracturs)

1. $\mathfrak{D} := \mathfrak{D}(X)$: algebra of invariant differential operators on X .
 $\mathfrak{U} := \mathfrak{U}(\mathfrak{g})$; $\mathfrak{U}^{\mathfrak{k}} := \mathfrak{Z}_{\mathfrak{k}}(\mathfrak{U})$; $(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}} := \mathfrak{U}\mathfrak{k} \cap \mathfrak{U}^{\mathfrak{k}}$. Will also need the center $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{U})$.
2. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$: Iwasawa decomposition (Group: $G = KAN$). Here $\mathfrak{a}$ is the Cartan subspace in $\mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{p}^+$. May identify $\mathfrak{S}(\mathfrak{a})$ with $\mathfrak{P}(\mathfrak{a}^*)$.
3. $\mathfrak{D} \cong \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$. This is the algebraic description of $\mathfrak{D}$.
4. $\gamma^0 : \mathfrak{D} \rightarrow \Lambda \subseteq \mathfrak{P}(\mathfrak{a}^*)$: the Harish-Chandra isomorphism into certain poly. ring.
 Good map! Essentially a symmetry-preserving projection.
5. !! By definition γ^0 is just a homom.; the fact that it is an isom. is non-trivial.

In context (fracturs)

1. $\mathfrak{D} := \mathfrak{D}(X)$: algebra of invariant differential operators on X .
 $\mathfrak{U} := \mathfrak{U}(\mathfrak{g})$; $\mathfrak{U}^{\mathfrak{k}} := \mathfrak{Z}_{\mathfrak{k}}(\mathfrak{U})$; $(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}} := \mathfrak{U}\mathfrak{k} \cap \mathfrak{U}^{\mathfrak{k}}$. Will also need the center $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{U})$.
2. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$: Iwasawa decomposition (Group: $G = KAN$). Here $\mathfrak{a}$ is the Cartan subspace in $\mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{p}^+$. May identify $\mathfrak{S}(\mathfrak{a})$ with $\mathfrak{P}(\mathfrak{a}^*)$.
3. $\mathfrak{D} \cong \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$. This is the algebraic description of $\mathfrak{D}$.
4. $\gamma^0 : \mathfrak{D} \rightarrow \Lambda \subseteq \mathfrak{P}(\mathfrak{a}^*)$: the Harish-Chandra isomorphism into certain poly. ring.
 Good map! Essentially a symmetry-preserving projection.
5. !! By definition γ^0 is just a homom.; the fact that it is an isom. is non-trivial.

In context (fra~~k~~turs)

1. $\mathfrak{D} := \mathfrak{D}(X)$: algebra of invariant differential operators on X .
 $\mathfrak{U} := \mathfrak{U}(\mathfrak{g})$; $\mathfrak{U}^{\mathfrak{k}} := \mathfrak{Z}_{\mathfrak{k}}(\mathfrak{U})$; $(\mathfrak{U}^{\mathfrak{k}})^{\mathfrak{k}} := \mathfrak{U}^{\mathfrak{k}} \cap \mathfrak{U}^{\mathfrak{k}}$. Will also need the center $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{U})$.
2. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$: Iwasawa decomposition (Group: $G = KAN$). Here $\mathfrak{a}$ is the Cartan subspace in $\mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{p}^+$. May identify $\mathfrak{S}(\mathfrak{a})$ with $\mathfrak{P}(\mathfrak{a}^*)$.
3. $\mathfrak{D} \cong \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}^{\mathfrak{k}})^{\mathfrak{k}}$. This is the algebraic description of $\mathfrak{D}$.
4. $\gamma^0 : \mathfrak{D} \rightarrow \Lambda \subseteq \mathfrak{P}(\mathfrak{a}^*)$: the Harish-Chandra isomorphism into certain poly. ring.
 Good map! Essentially a symmetry-preserving projection.
5. !! By definition γ^0 is just a homom.; the fact that it is an isom. is non-trivial. In particular, we see that

$\mathfrak{D}$ is a commutative algebra.

In context (fraktur)

1. $\mathfrak{D} := \mathfrak{D}(X)$: algebra of invariant differential operators on X .
 $\mathfrak{U} := \mathfrak{U}(\mathfrak{g})$; $\mathfrak{U}^{\mathfrak{k}} := \mathfrak{Z}_{\mathfrak{k}}(\mathfrak{U})$; $(\mathfrak{U}^{\mathfrak{k}})^{\mathfrak{k}} := \mathfrak{U}^{\mathfrak{k}} \cap \mathfrak{U}^{\mathfrak{k}}$. Will also need the center $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{U})$.
2. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$: Iwasawa decomposition (Group: $G = KAN$). Here $\mathfrak{a}$ is the Cartan subspace in $\mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{p}^+$. May identify $\mathfrak{S}(\mathfrak{a})$ with $\mathfrak{P}(\mathfrak{a}^*)$.
3. $\mathfrak{D} \cong \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}^{\mathfrak{k}})^{\mathfrak{k}}$. This is the algebraic description of $\mathfrak{D}$.
4. $\gamma^0 : \mathfrak{D} \rightarrow \Lambda \subseteq \mathfrak{P}(\mathfrak{a}^*)$: the Harish-Chandra isomorphism into certain poly. ring. Good map! Essentially a symmetry-preserving projection.
5. !! By definition γ^0 is just a homom.; the fact that it is an isom. is non-trivial. In particular, we see that

$\mathfrak{D}$ is a commutative algebra.

Discussed in Helgason's *Groups and Geometric Analysis, Chapter II*.

Schmid decomposition and Shimura operators

$$\mathcal{P}(n) := \{\lambda : \ell(\lambda) \leq n\}, \quad \mathcal{P}^d(n) := \{\lambda \in \mathcal{P}(n) : |\lambda| = d\}. \quad (n: \text{rank of } X = G/K; \dim \mathfrak{a}.)$$

Schmid decomposition and Shimura operators

$\mathcal{P}(n) := \{\lambda : \ell(\lambda) \leq n\}$, $\mathcal{P}^d(n) := \{\lambda \in \mathcal{P}(n) : |\lambda| = d\}$. (n : rank of $X = G/K$; $\dim \mathfrak{a}$.)

Schmid decomposition ([Sch70, FK90]) for $\mathfrak{k}$ -mods (multiplicity-free):

$$\mathfrak{S}^d(\mathfrak{p}^+) = \bigoplus_{\lambda \in \mathcal{P}^d(n)} W_\lambda, \quad \mathfrak{S}^d(\mathfrak{p}^-) = \bigoplus_{\lambda \in \mathcal{P}^d(n)} W_\lambda^*.$$

Here we choose a form to identify $\mathfrak{p}^-$ with $(\mathfrak{p}^+)^$. The highest weights are parametrized using the HC strongly orthogonal roots.*

Schmid decomposition and Shimura operators

$\mathcal{P}(n) := \{\lambda : \ell(\lambda) \leq n\}$, $\mathcal{P}^d(n) := \{\lambda \in \mathcal{P}(n) : |\lambda| = d\}$. (n : rank of $X = G/K$; $\dim \mathfrak{a}$.)
Schmid decomposition ([Sch70, FK90]) for $\mathfrak{k}$ -mods (multiplicity-free):

$$\mathfrak{S}^d(\mathfrak{p}^+) = \bigoplus_{\lambda \in \mathcal{P}^d(n)} W_\lambda, \quad \mathfrak{S}^d(\mathfrak{p}^-) = \bigoplus_{\lambda \in \mathcal{P}^d(n)} W_\lambda^*.$$

Here we choose a form to identify $\mathfrak{p}^-$ with $(\mathfrak{p}^+)^$. The highest weights are parametrized using the HC strongly orthogonal roots.*

Shimura Operators (those “1s”)

$$\begin{array}{c} \mathrm{End}_{\mathfrak{k}}(W_\lambda) \cong (W_\lambda^* \otimes W_\lambda)^{\mathfrak{k}} \hookrightarrow (\mathfrak{S}(\mathfrak{p}^-) \otimes \mathfrak{S}(\mathfrak{p}^+))^{\mathfrak{k}} \rightarrow \mathfrak{U}^{\mathfrak{k}} \rightarrow \mathfrak{D} \\ 1 \longmapsto \hspace{15em} \rightarrow D_\lambda \mapsto \mathscr{D}_\lambda \end{array}$$

Call $\mathscr{D}_\lambda$ the *Shimura operator associated with λ* .

A **basis** parameterized by **partitions**! Nice images of 1’s.

Classical result

Recall the H-C isomorphism $\gamma^0 : \mathfrak{D} \rightarrow \Lambda$. It captures/transport the Type BC symmetry (restricted root system $\rightarrow$ polynomials).

Classical result

Recall the H-C isomorphism $\gamma^0 : \mathfrak{D} \rightarrow \Lambda$. It captures/transport the Type BC symmetry (restricted root system $\rightarrow$ polynomials).

$$\begin{aligned} \mathrm{End}_{\mathfrak{k}}(W_{\lambda}) &\rightarrow \mathfrak{D} \xrightarrow{\gamma^0} \Lambda \\ 1 = \mathrm{Id}_{\lambda} &\longmapsto \gamma^0(\mathscr{D}_{\lambda}) \end{aligned}$$

Classical result

Recall the H-C isomorphism $\gamma^0 : \mathfrak{D} \rightarrow \Lambda$. It captures/transport the Type BC symmetry (restricted root system $\rightarrow$ polynomials).

$$\begin{aligned} \mathrm{End}_{\mathfrak{k}}(W_{\lambda}) &\rightarrow \mathfrak{D} \xrightarrow{\gamma^0} \Lambda \\ 1 = \mathrm{Id}_{\lambda} &\longmapsto \gamma^0(\mathscr{D}_{\lambda}) \end{aligned}$$

Theorem (Sahi & Zhang [SZ19])

We have $\gamma^0(\mathscr{D}_{\lambda}) = k_{\lambda} P_{\lambda}$ for some explicit $k_{\lambda} \neq 0$. Here P_{λ} 's are the Okounkov polynomials.

Classical result

Recall the H-C isomorphism $\gamma^0 : \mathfrak{D} \rightarrow \Lambda$. It captures/transport the Type BC symmetry (restricted root system $\rightarrow$ polynomials).

$$\begin{aligned} \text{End}_{\mathfrak{k}}(W_{\lambda}) &\rightarrow \mathfrak{D} \xrightarrow{\gamma^0} \Lambda \\ 1 = \text{Id}_{\lambda} &\longmapsto \gamma^0(\mathscr{D}_{\lambda}) \end{aligned}$$

Theorem (Sahi & Zhang [SZ19])

We have $\gamma^0(\mathscr{D}_{\lambda}) = k_{\lambda} P_{\lambda}$ for some explicit $k_{\lambda} \neq 0$. Here P_{λ} 's are the Okounkov polynomials.

Spectrum?

Let V_{μ} be the irreducible $\mathfrak{g}$ -module (of the same highest weight as W_{μ}). Then by the Cartan–Helgason Theorem, V_{μ} has a spherical vector $v^{\mathfrak{k}}$, i.e. $\mathfrak{k}.v^{\mathfrak{k}} = 0$. $D_{\lambda} \in \mathfrak{U}^{\mathfrak{k}} (\mathscr{D}_{\lambda} \in \mathfrak{D})$ acts on $v^{\mathfrak{k}}$ as $\gamma^0(\mathscr{D}_{\lambda})(\mu + \rho)$, hence the word **spectrum/eigenvalue**!

Lie Superalgebras

Definition (Lie object in $\mathbf{SVect}$)

A *Lie superalgebra* is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear map $[-, -]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ that respects parity grading and satisfies

1. $[X, Y] = -(-1)^{|X||Y|}[Y, X]$
2. $[[X, Y], Z] = [X, [Y, Z]] - (-1)^{|X||Y|}[Y, [X, Z]]$

Lie Superalgebras

Definition (Lie object in $\mathbf{SVect}$)

A *Lie superalgebra* is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear map $[-, -]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ that respects parity grading and satisfies

1. $[X, Y] = -(-1)^{|X||Y|}[Y, X]$
2. $[[X, Y], Z] = [X, [Y, Z]] - (-1)^{|X||Y|}[Y, [X, Z]]$

$\mathfrak{gl}(m|n)$:

$$\mathfrak{gl}_{\bar{0}}: \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \quad \mathfrak{gl}_{\bar{1}}: \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \quad [X, Y] := XY - (-1)^{|X||Y|}YX.$$

“Bad” news for super

- No Weyl’s theorem on complete reducibility
- Borels need not be conjugates
- *Isotropic* roots (length zero) exist
- No full Cartan–Helgason theorem on $\mathfrak{k}$ -spherical $\mathfrak{g}$ -mods

“Bad” news for super

- No Weyl’s theorem on complete reducibility
- Borels need not be conjugates
- *Isotropic* roots (length zero) exist
- No full Cartan–Helgason theorem on $\mathfrak{k}$ -spherical $\mathfrak{g}$ -mods

Nonetheless...

Supersymmetric Shimura operators

Fix $\mathfrak{g} = \mathfrak{gl}(2p|2q)$ and $\mathfrak{k} = \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q)$.

$\mathcal{H} = \mathcal{H}(p, q) := \{\lambda : \lambda_{p+1} \leq q\}$ (hook partitions). $\mathcal{H}^d := \{\lambda \in \mathcal{H} : |\lambda| = d\}$.

Super analog of the Schmid decomposition: ([CW01, SSS20])

$$\mathfrak{S}^d(\mathfrak{p}^+) = \bigoplus_{\lambda \in \mathcal{H}^d} W_\lambda, \quad \mathfrak{S}^d(\mathfrak{p}^-) = \bigoplus_{\lambda \in \mathcal{H}^d} W_\lambda^*.$$

W_λ are of Type M, and $\dim \text{End}_{\mathfrak{k}}(W_\lambda) = 1$. (Super version of Schur's Lemma may include the parity twist.) Set $\mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}^{\mathfrak{k}})^{\mathfrak{k}}$.

Supersymmetric Shimura operators

Fix $\mathfrak{g} = \mathfrak{gl}(2p|2q)$ and $\mathfrak{k} = \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q)$.

$\mathcal{H} = \mathcal{H}(p, q) := \{\lambda : \lambda_{p+1} \leq q\}$ (hook partitions). $\mathcal{H}^d := \{\lambda \in \mathcal{H} : |\lambda| = d\}$.

Super analog of the Schmid decomposition: ([CW01, SSS20])

$$\mathfrak{S}^d(\mathfrak{p}^+) = \bigoplus_{\lambda \in \mathcal{H}^d} W_\lambda, \quad \mathfrak{S}^d(\mathfrak{p}^-) = \bigoplus_{\lambda \in \mathcal{H}^d} W_\lambda^*.$$

W_λ are of Type M, and $\dim \text{End}_{\mathfrak{k}}(W_\lambda) = 1$. (Super version of Schur's Lemma may include the parity twist.) Set $\mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}^{\mathfrak{k}})^{\mathfrak{k}}$.

Supersymmetric Shimura Operators [super cousins]

$$\begin{array}{ccccccc} \text{End}_{\mathfrak{k}}(W_\lambda) & \cong & (W_\lambda^* \otimes W_\lambda)^{\mathfrak{k}} & \hookrightarrow & (\mathfrak{S}(\mathfrak{p}^-) \otimes \mathfrak{S}(\mathfrak{p}^+))^{\mathfrak{k}} & \rightarrow & \mathfrak{U}^{\mathfrak{k}} \rightarrow \mathfrak{D} \\ & & 1 & \xlongequal{\hspace{1.5cm}} & & \rightarrow & D_\lambda \mapsto \mathscr{D}_\lambda \end{array}$$

Results

And yes, the super Harish-Chandra **isomorphism** $\gamma^0 : \mathfrak{D} \rightarrow \Lambda^0(\mathfrak{a}^*) \subseteq \mathfrak{P}(\mathfrak{a}^*)$ exists.

Results

And yes, the super Harish-Chandra **isomorphism** $\gamma^0 : \mathfrak{D} \rightarrow \Lambda^0(\mathfrak{a}^*) \subseteq \mathfrak{P}(\mathfrak{a}^*)$ exists.
Likewise

1. $\mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$ is regarded as the space of diff. op.'s associated with $(\mathfrak{g}, \mathfrak{k})$

Results

And yes, the **super Harish-Chandra isomorphism** $\gamma^0 : \mathfrak{D} \rightarrow \Lambda^0(\mathfrak{a}^*) \subseteq \mathfrak{P}(\mathfrak{a}^*)$ exists.
Likewise

1. $\mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$ is regarded as the space of diff. op.'s associated with $(\mathfrak{g}, \mathfrak{k})$
2. $\Lambda^0(\mathfrak{a}^*)$ is the ring of even supersymmetric polynomials on $\mathfrak{a}^*$

Results

And yes, the **super Harish-Chandra isomorphism** $\gamma^0 : \mathfrak{D} \rightarrow \Lambda^0(\mathfrak{a}^*) \subseteq \mathfrak{P}(\mathfrak{a}^*)$ exists.
Likewise

1. $\mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$ is regarded as the space of diff. op.'s associated with $(\mathfrak{g}, \mathfrak{k})$
2. $\Lambda^0(\mathfrak{a}^*)$ is the ring of even supersymmetric polynomials on $\mathfrak{a}^*$

It is again non-trivial that γ^0 is an isom.

Results

And yes, the **super Harish-Chandra isomorphism** $\gamma^0 : \mathfrak{D} \rightarrow \Lambda^0(\mathfrak{a}^*) \subseteq \mathfrak{P}(\mathfrak{a}^*)$ exists.
Likewise

1. $\mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$ is regarded as the space of diff. op.'s associated with $(\mathfrak{g}, \mathfrak{k})$
2. $\Lambda^0(\mathfrak{a}^*)$ is the ring of even supersymmetric polynomials on $\mathfrak{a}^*$

It is again non-trivial that γ^0 is an isom.

Theorem A (Sahi–Z. [SZ25])

We have $\gamma^0(\mathscr{D}_\mu) = k_\mu J_\mu$ where $k_\mu = (-1)^{|\mu|} \prod_{(i,j) \in \mu} (\mu_i - j + \mu'_j - i + 1)$. Here J_μ are Sergeev–Veselov interpolation polynomials.

The main thing is to show the vanishing properties defining those SV polys.

Classically: Can directly use f.d. spherical irrep.s to capture the action of $\mathscr{D}_\mu$

Super: We don't have “enough”!

A nice diagram

The image of 1 under a good map is good.

A nice diagram

The image of 1 under a good map is good.

The image of 1 under a super good map is super good.

A nice diagram

The image of 1 under a good map is good.

The image of 1 under a super good map is super good.

Good maps: Harish-Chandra isom γ^0 ; Images: certain symmetric polynomials.

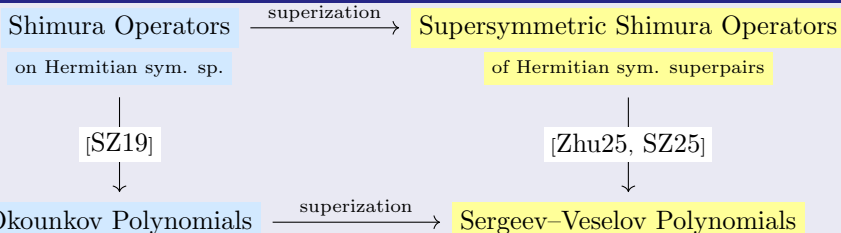
A nice diagram

The image of 1 under a good map is good.

The image of 1 under a super good map is super good.

Good maps: Harish-Chandra isom γ^0 ; Images: certain symmetric polynomials.

Shimura operators and interpolation polynomials



Recap of the Shimura Story

1. Classical and super alike, we may study the Shimura operators *explicitly defined* via rep theory (Schmid decompositions).

Recap of the Shimura Story

1. Classical and super alike, we may study the Shimura operators *explicitly defined* via rep theory (Schmid decompositions).
2. They live in $\mathfrak{D}$ (the space of diff. op.'s) which is commutative.

Recap of the Shimura Story

1. Classical and super alike, we may study the Shimura operators *explicitly defined* via rep theory (Schmid decompositions).
2. They live in $\mathfrak{D}$ (the space of diff. op.'s) which is commutative.
3. There are enough of them, generating $\mathfrak{D}$

Recap of the Shimura Story

1. Classical and super alike, we may study the Shimura operators *explicitly defined* via rep theory (Schmid decompositions).
2. They live in $\mathfrak{D}$ (the space of diff. op.'s) which is commutative.
3. There are enough of them, generating $\mathfrak{D}$
4. (We also understand their spectra in terms of symmetric functions.)

Recap of the Shimura Story

1. Classical and super alike, we may study the Shimura operators *explicitly defined* via rep theory (Schmid decompositions).
2. They live in $\mathfrak{D}$ (the space of diff. op.'s) which is commutative.
3. There are enough of them, generating $\mathfrak{D}$
4. (We also understand their spectra in terms of symmetric functions.)

Recap of the Shimura Story

1. Classical and super alike, we may study the Shimura operators *explicitly defined* via rep theory (Schmid decompositions).
2. They live in $\mathfrak{D}$ (the space of diff. op.'s) which is commutative.
3. There are enough of them, generating $\mathfrak{D}$
4. (We also understand their spectra in terms of symmetric functions.)

Scope

$(\mathfrak{g}, \mathfrak{k})$ pairs admitting an Iwasawa decomposition and having analogous Schmid decomp.

(Kantor–Koecher–Tits correspondence with Jordan superalgebras)

Physics

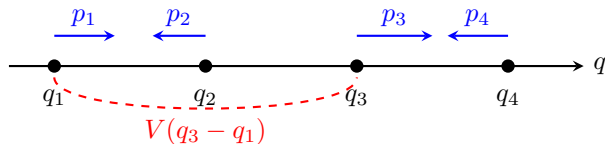
1. Consider n particles on a line; each has momentum, and interacting with each other through pairwise potentials.

Physics

1. Consider n particles on a line; each has momentum, and interacting with each other through pairwise potentials.

Physics

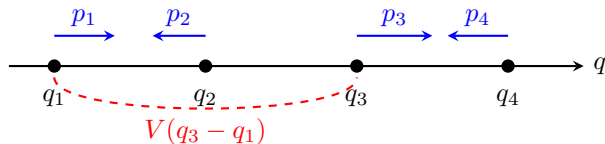
1. Consider n particles on a line; each has momentum, and interacting with each other through pairwise potentials.



2. The Hamiltonian $\mathcal{H}$ = kinetic energy (quadratic in the momenta) + potential energy (depending on positions)

Physics

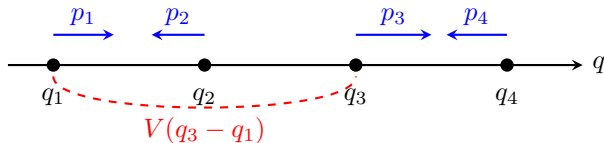
1. Consider n particles on a line; each has momentum, and interacting with each other through pairwise potentials.



2. The Hamiltonian $\mathcal{H}$ = kinetic energy (quadratic in the momenta) + potential energy (depending on positions)
3. Pairwise interaction $\rightsquigarrow$ Type A root system. Can ask for general root system configuration.

Physics

1. Consider n particles on a line; each has momentum, and interacting with each other through pairwise potentials.



2. The Hamiltonian $\mathcal{H}$ = kinetic energy (quadratic in the momenta) + potential energy (depending on positions)
3. Pairwise interaction $\rightsquigarrow$ Type A root system. Can ask for general root system configuration.
4. We work with *quantum setting*: momenta = diff. op.'s, and the $\mathcal{H}$ = CMS diff. op. of ord. 2.

Integrability means...?

- **Classical complete *integrability*:** independent deg.-of-freedom many *integrals* of motion, including the Hamiltonian, which pairwise Poisson $\{-, -\}$ -commute.
- **Quantum complete *integrability* (in our setting):** deg.-of-freedom many pairwise commuting diff. op.'s (so $[-, -] = 0$), including the Hamiltonian, whose principal symbols are algebraically independent.

Integrability means...?

- **Classical complete *integrability*:** independent deg.-of-freedom many *integrals* of motion, including the Hamiltonian, which pairwise Poisson $\{-, -\}$ -commute.
- **Quantum complete *integrability* (in our setting):** deg.-of-freedom many pairwise commuting diff. op.'s (so $[-, -] = 0$), including the Hamiltonian, whose principal symbols are algebraically independent.

Parameters	Options
Potential V	Rational (I); trigonometric (II)/hyperbolic (III); elliptic (IV)
Configuration	Root systems ; deformed root systems ; can be even more general...
Our focus	Trigonometric/hyperbolic type BC and its super analogs

CMS Operator

In the quantum setting, the **gauged Hamiltonian** with hyperbolic potential is

$$\mathcal{L} = \Delta_{\mathfrak{a}} + \sum_{\alpha \in \Sigma^+} m_{\alpha} \coth(\alpha(x)) \partial_{\alpha}.$$

- Σ : root system, possibly non-reduced.
- m_{α} : root multiplicity, which is W -inv. (Weyl group of Σ .)
- Gauge: conjugation by an explicit function that makes our lives easier...

CMS Operator

In the quantum setting, the **gauged Hamiltonian** with hyperbolic potential is

$$\mathcal{L} = \Delta_{\mathfrak{a}} + \sum_{\alpha \in \Sigma^+} m_{\alpha} \coth(\alpha(x)) \partial_{\alpha}.$$

- Σ : root system, possibly non-reduced.
- m_{α} : root multiplicity, which is W -inv. (Weyl group of Σ .)
- Gauge: conjugation by an explicit function that makes our lives easier...

For type BC_n , quantum complete integrability = constructing **pairwise commuting** differential operators of orders $2, 4, \dots, 2n$ including $\mathcal{L}$, whose principal symbols are algebraically independent. (Evenness comes from BC symmetry.)

CMS Operator

In the quantum setting, the **gauged Hamiltonian** with hyperbolic potential is

$$\mathcal{L} = \Delta_{\mathfrak{a}} + \sum_{\alpha \in \Sigma^+} m_{\alpha} \coth(\alpha(x)) \partial_{\alpha}.$$

- Σ : root system, possibly non-reduced.
- m_{α} : root multiplicity, which is W -inv. (Weyl group of Σ .)
- Gauge: conjugation by an explicit function that makes our lives easier...

For type BC_n , quantum complete integrability = constructing **pairwise commuting** differential operators of orders $2, 4, \dots, 2n$ including $\mathcal{L}$, whose principal symbols are algebraically independent. (Evenness comes from BC symmetry.)

In the non-super setting, Dunkl-type operators (which involve reflections) provide one construction.

Radial Part

Now in geometry, we have *radial parts* of a diff. op.

Radial Part

Now in geometry, we have *radial parts* of a diff. op.

Radial Part

V : a manifold, H : a Lie group acting on V , $\mathfrak{h}$: its Lie alg., W : a submanifold so that

$$V_x = (Hx)_x \oplus W_x, \text{ for each } x \in W \quad [\text{Transversality}]$$

Radial Part

Now in geometry, we have *radial parts* of a diff. op.

Radial Part

V : a manifold, H : a Lie group acting on V , $\mathfrak{h}$: its Lie alg., W : a submanifold so that

$$V_x = (Hx)_x \oplus W_x, \text{ for each } x \in W \quad [\text{Transversality}]$$

Then $\forall$ diff. op. D on V , $\exists!$ $\text{Rad}(D)$ on W (the *radial part* of D) so that $\forall$ (locally) H -invariant f

$$(Df)|_W = \text{Rad}(D)(f|_W)$$

Radial Part

Now in geometry, we have *radial parts* of a diff. op.

Radial Part

V : a manifold, H : a Lie group acting on V , $\mathfrak{h}$: its Lie alg., W : a submanifold so that

$$V_x = (Hx)_x \oplus W_x, \text{ for each } x \in W \quad [\text{Transversality}]$$

Then $\forall$ diff. op. D on V , $\exists!$ $\text{Rad}(D)$ on W (the *radial part* of D) so that $\forall$ (locally) H -invariant f

$$(Df)|_W = \text{Rad}(D)(f|_W)$$

In our context

$V = X = G/K$, $H = K$, $\mathfrak{h} = \mathfrak{k}$, $W = A^+.eK$, where $A^+ = \exp(\mathfrak{a}^+)$ of a Weyl chamber.
The tangent space is really $\mathfrak{a}$ in the Iwasawa decomp.

Upshot

Using “Helgason’s algorithm” [Hel00, Proposition 5.23, Ch. II] we can explicitly write down $\text{Rad}(D)$.

Shimura op.s’ radial part

$\text{Rad}(\mathcal{D}_{\square}) \propto \mathcal{L}$ with some natural identification.

Upshot

Using “Helgason’s algorithm” [Hel00, Proposition 5.23, Ch. II] we can explicitly write down $\text{Rad}(D)$.

Shimura op.s’ radial part

$\text{Rad}(\mathcal{D}_{\square}) \propto \mathcal{L}$ with some natural identification.

Explicit calculation

Fix a bilinear form to identify $(\mathfrak{p}^+)^*$ and $\mathfrak{p}^-$ and pick bases $\{x_i^{\pm}\}$ for $\mathfrak{p}^{\pm}$ dual to each other. Then $\mathcal{D}_{\square}$ has the following representative in $\mathfrak{U}^{\mathfrak{t}}$

$$\sum x_i^- x_i^+.$$

The Casimir $\Omega_{\mathfrak{g}} = \underbrace{\sum_a Z_a Z^a}_{\Omega_{\mathfrak{k}}} + \sum_i (x_i^+ x_i^- + x_i^- x_i^+)$ is congruent to $2\mathcal{D}_{\square} \bmod \mathfrak{U}^{\mathfrak{k}}$. But $\text{Rad}(\Omega_{\mathfrak{g}}) = \mathcal{L}$.

$$\implies \text{Rad}(\mathcal{D}_{\square}) = 1/2\mathcal{L}.$$

Upshot

A routine check verifies that taking radial parts is a homom.:

$$\mathrm{Rad}(DE) = \mathrm{Rad}(D)\mathrm{Rad}(E)$$

Upshot

A routine check verifies that taking radial parts is a homom.:

$$\text{Rad}(DE) = \text{Rad}(D)\text{Rad}(E)$$

Here comes the bridge between two stories: Now using the Shimura story, since we have $\mathcal{D}_\mu$ (ord. $2|\mu|$) for all μ , the following op.'s

$$\mathcal{D}_\square, \quad \mathcal{D}_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}, \quad \mathcal{D}_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}}, \dots, \mathcal{D}_{\begin{smallmatrix} \square \\ \vdots \\ \square \end{smallmatrix}}$$

are algebraically independent.

Quantum Integrability of $\mathcal{L}$

All $\text{Rad}(\mathcal{D}_\mu)$ commute; in particular $\mathcal{L} = 2\text{Rad}(\mathcal{D}_\square)$. Thus $\mathcal{L}$ is quantum integrable.

Deformed CMS

The **deformed CMS operator in radial gauge** is

$$\mathcal{L} = \Delta_{\mathfrak{a}} + \sum_{\alpha \in \Sigma^+} m_{\alpha} \coth(\alpha(t)) \partial_{\alpha}.$$

Deformed CMS

The **deformed CMS operator in radial gauge** is

$$\mathcal{L} = \Delta_{\mathfrak{a}} + \sum_{\alpha \in \Sigma^+} m_{\alpha} \coth(\alpha(t)) \partial_{\alpha}.$$

Deformed in the sense that

- Multiplicities m_{α} are now

$$m_{\alpha} = \dim(\mathfrak{g}_{\alpha})_{\bar{0}} - \dim(\mathfrak{g}_{\alpha})_{\bar{1}}.$$

- $\Delta_{\mathfrak{a}}$ and ∂_{α} use the invariant bilinear form, which need not be positive definite.

The $p + q$ coordinates remain ordinary commuting variables.

What should a **super** radial part be? The usual **geometry** doesn't quite work....

Geometry to algebra

Geometry	Algebra (classical and super)
Symmetric space $X = G/K$, with $o = eK$.	Symmetric pair $(\mathfrak{g}, \mathfrak{k})$; write $\mathfrak{U} = \mathfrak{U}(\mathfrak{g})$.
K -inv. functions on X : $f \in C^\infty(X)^K$. Equivalently, bi- K -inv. functions F on G : $F(kgk') = F(g)$.	Bi- $\mathfrak{k}$ -inv. linear functionals: $\mathcal{Z}_X = \{f \in \mathfrak{U}^* : f(\mathfrak{k}\mathfrak{U} + \mathfrak{U}\mathfrak{k}) = 0\}.$ [SV17]
Inv. diff. op.'s $\mathfrak{D}(X)$ preserves $C^\infty(X)^K$.	For even $D \in \mathfrak{U}^\mathfrak{k}$ set $(R_D f)(u) = f(uD)$. This action preserves $\mathcal{Z}_X$ and descends to $\mathfrak{D} = \mathfrak{U}^\mathfrak{k}/(\mathfrak{U}\mathfrak{k})^\mathfrak{k}$.
Restriction to the positive chamber: $A^+ = \exp(\mathfrak{a}^+), \quad \text{res}(f) = f _{A^+ \cdot o}.$ The restricted functions lie in $C^\infty(A^+ \cdot o)$.	Choose a basis $A_1, \dots, A_r$ of $\mathfrak{a}$: $j(f) := \hat{f}(t) = \sum_{\beta \in \mathbb{N}^r} \frac{f(A^\beta)}{\beta!} t^\beta.$ Set $\mathcal{A}_X = j(\mathcal{Z}_X) \subseteq \mathbb{C}[[t_1, \dots, t_r]]$. Here $z^\beta = z_1^{\beta_1} \dots z_r^{\beta_r}$

Thus we now consider (honest) diff. op.'s on formal power series!

Geometry to algebra

Geometry	Algebra (classical and super)
$ \begin{array}{ccc} C^\infty(X)^K & \xrightarrow{D} & C^\infty(X)^K \\ \text{res} \downarrow & & \downarrow \text{res} \\ C^\infty(A^+ \cdot o) & \xrightarrow{\text{Rad}(D)} & C^\infty(A^+ \cdot o) \end{array} $	$ \begin{array}{ccc} \mathcal{L}_X & \xrightarrow{R_D} & \mathcal{L}_X \\ j \downarrow & & \downarrow j \\ \mathcal{A}_X & \xrightarrow{\text{Rad}_X(D)} & \mathcal{A}_X \end{array} $
The radial part is a differential operator satisfying $\text{res} \circ D = \text{Rad}(D) \circ \text{res}.$ It respects products: $\text{Rad}(DE) = \text{Rad}(D)\text{Rad}(E).$	Once j is proved injective, define $\text{Rad}_X(D) = j \circ R_D \circ j^{-1}.$ This is an <i>algebra homomorphism</i> (for free!) into $\text{End}(\mathcal{A}_X)$.

Results

Proposition

Let $\mathcal{R}$ denote certain localized coefficient algebra, equipped with $\partial_{t_1}, \dots, \partial_{t_r}$. For each $u \in \mathfrak{U}$ there exists a diff. op. P_u with $\mathcal{R}$ coeff. such that

$$P_u(j(f)) = \sum_{\beta} \frac{f(A^{\beta}u)}{\beta!} t^{\beta}.$$

Proposition

The restriction map $j : \mathcal{Z}_X \rightarrow \mathcal{A}_X$ is injective.

Results

Proposition

Let $\mathcal{R}$ denote certain localized coefficient algebra, equipped with $\partial_{t_1}, \dots, \partial_{t_r}$. For each $u \in \mathfrak{U}$ there exists a diff. op. P_u with $\mathcal{R}$ coeff. such that

$$P_u(j(f)) = \sum_{\beta} \frac{f(A^{\beta}u)}{\beta!} t^{\beta}.$$

Proposition

The restriction map $j : \mathcal{Z}_X \rightarrow \mathcal{A}_X$ is injective.

Proof.

Suppose $j(f) = 0$. Then for all $u \in \mathfrak{U}$ we have $P_u(j(f)) = 0$.

Results

Proposition

Let $\mathcal{R}$ denote certain localized coefficient algebra, equipped with $\partial_{t_1}, \dots, \partial_{t_r}$. For each $u \in \mathfrak{U}$ there exists a diff. op. P_u with $\mathcal{R}$ coeff. such that

$$P_u(j(f)) = \sum_{\beta} \frac{f(A^{\beta}u)}{\beta!} t^{\beta}.$$

Proposition

The restriction map $j : \mathcal{Z}_X \rightarrow \mathcal{A}_X$ is injective.

Proof.

Suppose $j(f) = 0$. Then for all $u \in \mathfrak{U}$ we have $P_u(j(f)) = 0$. But this implies $\frac{f(A^{\beta}u)}{\beta!} t^{\beta} = 0$ for all β . In particular when $\beta = 0$, we have $f(u) = 0$. □

Results

A similar computation shows that

$$\mathrm{Rad}_X(\mathcal{D}_{\square}) = \frac{1}{2}\mathcal{L}$$

as operators on $\mathcal{A}_X$.

Currently working on...

P_u 's uniqueness; algebraic independence of $\mathrm{Rad}_X(\mathcal{D}_{\mu})$.

With the above Rad_X in mind, using the Shimura theory, one can obtain a commuting family of diff. op.'s of the *algebraic radial parts* of the Shimura operators. This would show the quantum integrability of $\mathcal{L}$.

Scope and Disclosure

We expect this to work for all the Shimura pairs.

Scope and Disclosure

We expect this to work for all the Shimura pairs.

But we should also point out the limitation:

1. These pairs give rise to BC -type symmetry, with root multiplicities determined by the pair.

Scope and Disclosure

We expect this to work for all the Shimura pairs.

But we should also point out the limitation:

1. These pairs give rise to BC -type symmetry, with root multiplicities determined by the pair. In the classical setting, Heckman–Opdam theory allows arbitrary multiplicity parameters [HO87]. Can our construction admit a similar extension?
2. The Shimura theory provides an alternative proof of known integrability results. We hope to extend it to new settings. (Other types of configurations/potentials etc.)

Scope and Disclosure

We expect this to work for all the Shimura pairs.

But we should also point out the limitation:

1. These pairs give rise to BC -type symmetry, with root multiplicities determined by the pair. In the classical setting, Heckman–Opdam theory allows arbitrary multiplicity parameters [HO87]. Can our construction admit a similar extension?
2. The Shimura theory provides an alternative proof of known integrability results. We hope to extend it to new settings. (Other types of configurations/potentials etc.)

All problems and statements are human proposed. For the recent progress, AI suggested a paper [SV17] we overlooked that essentially reveals the algebraic construction of radial parts; AI also helps with computations and certain proof ideas.

All results are human checked.

Thank you!

-  Alexander Alldridge and Sebastian Schmittner, *Spherical representations of Lie supergroups*, J. Funct. Anal. **268** (2015), no. 6, 1403–1453. MR 3306354
-  Shun-Jen Cheng and Weiqiang Wang, *Howe duality for Lie superalgebras*, Compositio Math. **128** (2001), no. 1, 55–94. MR 1847665
-  J. Faraut and A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), no. 1, 64–89. MR 1033914
-  Sigurdur Helgason, *Groups and geometric analysis*, Mathematical Surveys and Monographs, vol. 83, American Mathematical Society, Providence, RI, 2000, Integral geometry, invariant differential operators, and spherical functions, Corrected reprint of the 1984 original. MR 1790156
-  G. J. Heckman and E. M. Opdam, *Root systems and hypergeometric functions. I*, Compositio Math. **64** (1987), no. 3, 329–352. MR 918416
-  Wilfried Schmid, *Die Randwerte holomorpher Funktionen auf hermiteschen Räumen*, Invent. Math. **9** (1969/70), 61–80. MR 259164
-  Goro Shimura, *Invariant differential operators on Hermitian symmetric spaces*, Ann. of Math. (2) **132** (1990), no. 2, 237–272. MR 1070598
-  Siddhartha Sahi, Hadi Salmasian, and Vera Serganova, *The Capelli eigenvalue problem for Lie superalgebras*, Math. Z. **294** (2020), no. 1-2, 359–395. MR 4054814
-  John R. Stembridge, *A characterization of supersymmetric polynomials*, J. Algebra **95** (1985), no. 2, 439–444. MR 801279
-  A. N. Sergeev and A. P. Veselov, *Generalised discriminants, deformed Calogero-Moser-Sutherland operators and super-Jack polynomials*, Adv. Math. **192** (2005), no. 2, 341–375. MR 2128703
-  ———, *Symmetric Lie superalgebras and deformed quantum Calogero-Moser problems*, Adv. Math. **304** (2017), 728–768. MR 3558220
-  Siddhartha Sahi and Genkai Zhang, *Positivity of Shimura operators*, Math. Res. Lett. **26** (2019), no. 2, 587–626. MR 3999556



Siddhartha Sahi and Songhao Zhu, *Eigenvalues of supersymmetric Shimura operators and interpolation polynomials*, Adv. Math. **481** (2025), Paper No. 110547, 32. MR 4963163



Songhao Zhu, *Shimura operators for certain Hermitian symmetric superpairs*, J. Lie Theory **35** (2025), no. 2, 377–410.

Set up

Fix $\mathfrak{g} = \mathfrak{gl}(2p|2q)$ and $\mathfrak{k} = \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q)$. Embed $\mathfrak{k}$ into $\mathfrak{g}$:

$$\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right), \left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right) \right) \mapsto \left(\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ \hline C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{array} \right)$$

Here $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) consists of matrices with non-zero entries only in the upper right (resp. bottom left) sub-blocks in each of the four blocks.

Set up

Fix $\mathfrak{g} = \mathfrak{gl}(2p|2q)$ and $\mathfrak{k} = \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q)$. Embed $\mathfrak{k}$ into $\mathfrak{g}$:

$$\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right), \left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right) \right) \mapsto \left(\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ \hline C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{array} \right)$$

Here $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) consists of matrices with non-zero entries only in the upper right (resp. bottom left) sub-blocks in each of the four blocks.

Technical details:

1. Let $J := \frac{1}{2} \operatorname{diag}(I, -I, I, -I)$, and $\theta := \operatorname{Ad} \exp(i\pi J)$. Then θ has fixed point subalgebra $\mathfrak{k}$.
2. The standard diagonal Cartan is denoted as $\mathfrak{t}$ (in both $\mathfrak{g}$ and $\mathfrak{k}$, “max. compact”)
3. Fix a θ -stable, maximally split Cartan $\mathfrak{h}$ containing $\mathfrak{a}$, a maximal toral subalgebra in $\mathfrak{p}_{\overline{0}}$ (for Iwasawa decomp.).

Supersymmetric polynomials

For polynomials, supersymmetry = symmetry + translational invariance.

Supersymmetric polynomials

For polynomials, supersymmetry = symmetry + translational invariance.
More precisely, in the Type BC situation:

Definition

Let $\{\epsilon_i\}_{i=1}^p \cup \{\delta_j\}_{j=1}^q$ be the standard basis for $V = \mathbb{C}^{p+q}$. Denote the coordinate functions for this basis as x_i and y_j . A polynomial $f \in \mathfrak{P}(V) = \mathbb{C}[x_i, y_j]$ is said to be *even supersymmetric* if :

- (i) f is symmetric in x_i and y_j separately and invariant under sign changes of x_i and y_j ;
- (ii) $f(X + \epsilon_i - \delta_j) = f(X)$ if $x_i + y_j = 0$ for $i = 1, \dots, p, j = 1, \dots, q$.

We denote the subring of even supersymmetric polynomials as $\Lambda^0(V)$.

!! (i) $\iff$ invariance under $(S_p \ltimes \mathbb{Z}_2^n) \times (S_q \ltimes \mathbb{Z}_2^n)$ but (ii) is no longer group invariance (not even linear).

In fact, one may use a suitable **Weyl groupoid** action to capture (i & ii).

Main Results

1. $\gamma : \mathfrak{Z} \rightarrow \mathfrak{P}(\mathfrak{h}^*)$: the usual Harish-Chandra isomorphism on the center $\mathfrak{Z} \subseteq \mathfrak{U}$; This is the map that captures how the center acts on any highest weight module (generated by one vec with nice properties)
2. $\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t}_+$: Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$;
3. **Res**: the restriction map induced from the decomposition $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}_+$.

Main Results

1. $\gamma : \mathfrak{Z} \rightarrow \mathfrak{P}(\mathfrak{h}^*)$: the usual Harish-Chandra isomorphism on the center $\mathfrak{Z} \subseteq \mathfrak{U}$; This is the map that captures how the center acts on any highest weight module (generated by one vec with nice properties)
2. $\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t}_+$: Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$;
3. Res : the restriction map induced from the decomposition $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}_+$.

We have $\pi : \mathfrak{Z} \hookrightarrow \mathfrak{U}^{\mathfrak{k}} \twoheadrightarrow \mathfrak{D} := \mathfrak{U}^{\mathfrak{k}} / (\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$.

$$\begin{array}{ccc}
 \mathfrak{Z} & \xrightarrow{\pi} & \mathfrak{D} \\
 \gamma \downarrow & & \downarrow \gamma^0 \\
 \Lambda(\mathfrak{h}^*) & \xrightarrow{\text{Res}} & \Lambda^0(\mathfrak{a}^*)
 \end{array}$$

Main point: check $\gamma^0 \circ \pi = \text{Res} \circ \gamma$ on a Zariski dense set!

Main Results

1. $\gamma : \mathfrak{Z} \rightarrow \mathfrak{P}(\mathfrak{h}^*)$: the usual Harish-Chandra isomorphism on the center $\mathfrak{Z} \subseteq \mathfrak{U}$; This is the map that captures how the center acts on any highest weight module (generated by one vec with nice properties)
2. $\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t}_+$: Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$;
3. Res : the restriction map induced from the decomposition $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}_+$.

We have $\pi : \mathfrak{Z} \hookrightarrow \mathfrak{U}^{\mathfrak{k}} \twoheadrightarrow \mathfrak{D} := \mathfrak{U}^{\mathfrak{k}} / (\mathfrak{U}^{\mathfrak{k}})^{\mathfrak{k}}$.

$$\begin{array}{ccc}
 \mathfrak{Z} & \xrightarrow{\pi} & \mathfrak{D} \\
 \gamma \downarrow & & \downarrow \gamma^0 \\
 \Lambda(\mathfrak{h}^*) & \xrightarrow{\text{Res}} & \Lambda^0(\mathfrak{a}^*)
 \end{array}$$

Main point: check $\gamma^0 \circ \pi = \text{Res} \circ \gamma$ on a Zariski dense set! The set of highest weights of f.d. irreducible & spherical $\mathfrak{g}$ -mods.

Main Results

$$\begin{array}{ccc} Z_\mu & \xrightarrow{\pi} & \mathcal{D}_\mu \\ \gamma \downarrow & & \downarrow \gamma^0 \\ \gamma(Z_\mu) & \xrightarrow{\text{Res}} & \gamma^0(\mathcal{D}_\mu) \end{array}$$

Theorem B (Sahi–Z. [SZ25])

The map π is surjective. In particular, there exists $Z_\mu \in \mathfrak{Z}$ such that $\pi(Z_\mu) = \mathcal{D}_\mu$. (So $\mathcal{D}_\mu = \pi(D_\mu)$ can be captured by some central element!)

Let $I_\lambda := \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda$ be the generalized Verma module for $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$.

Theorem C (Sahi–Z. [SZ25])

The central element Z_μ acts on I_λ by 0 when $\lambda \not\geq \mu$.

Main Results

Theorem A

We have $\gamma^0(\mathscr{D}_\mu) = k_\mu J_\mu$ where $k_\mu = (-1)^{|\mu|} \prod_{(i,j) \in \mu} (\mu_i - j + \mu'_j - i + 1)$.

Sketch of the proof of Theorem A.

1. Commutative diag. $\implies \gamma^0(\pi(Z_\mu)) = \text{Res}(\gamma(Z_\mu))$. The LHS is just $\gamma^0(\mathscr{D}_\mu)$.

Main Results

Theorem A

We have $\gamma^0(\mathcal{D}_\mu) = k_\mu J_\mu$ where $k_\mu = (-1)^{|\mu|} \prod_{(i,j) \in \mu} (\mu_i - j + \mu'_j - i + 1)$.

Sketch of the proof of Theorem A.

1. Commutative diag. $\implies \gamma^0(\pi(Z_\mu)) = \text{Res}(\gamma(Z_\mu))$. The LHS is just $\gamma^0(\mathcal{D}_\mu)$.
2. $\gamma^0(\mathcal{D}_\mu)(\overline{\lambda} + \rho) = \text{Res} \circ \gamma(Z_\mu)(\overline{\lambda} + \rho)$

Main Results

Theorem A

We have $\gamma^0(\mathscr{D}_\mu) = k_\mu J_\mu$ where $k_\mu = (-1)^{|\mu|} \prod_{(i,j) \in \mu} (\mu_i - j + \mu'_j - i + 1)$.

Sketch of the proof of Theorem A.

1. Commutative diag. $\implies \gamma^0(\pi(Z_\mu)) = \text{Res}(\gamma(Z_\mu))$. The LHS is just $\gamma^0(\mathscr{D}_\mu)$.
2. $\gamma^0(\mathscr{D}_\mu)(\bar{\lambda} + \rho) = \text{Res} \circ \gamma(Z_\mu)(\bar{\lambda} + \rho)$
3. By Theorem C, Z_μ acts by 0 on I_λ for $\lambda \neq \mu$. But $\mathbb{Z}$ acts on a cyclic module exactly by γ . Thus $\gamma^0(\mathscr{D}_\mu)(\bar{\lambda} + \rho) = 0$, for all $\lambda \neq \mu$, $\lambda \in \mathcal{H}$.

Main Results

Theorem A

We have $\gamma^0(\mathscr{D}_\mu) = k_\mu J_\mu$ where $k_\mu = (-1)^{|\mu|} \prod_{(i,j) \in \mu} (\mu_i - j + \mu'_j - i + 1)$.

Sketch of the proof of Theorem A.

1. Commutative diag. $\implies \gamma^0(\pi(Z_\mu)) = \text{Res}(\gamma(Z_\mu))$. The LHS is just $\gamma^0(\mathscr{D}_\mu)$.
2. $\gamma^0(\mathscr{D}_\mu)(\bar{\lambda} + \rho) = \text{Res} \circ \gamma(Z_\mu)(\bar{\lambda} + \rho)$
3. By Theorem C, Z_μ acts by 0 on I_λ for $\lambda \not\preceq \mu$. But $\mathbb{Z}$ acts on a cyclic module exactly by γ . Thus $\gamma^0(\mathscr{D}_\mu)(\bar{\lambda} + \rho) = 0$, for all $\lambda \not\preceq \mu$, $\lambda \in \mathcal{H}$.
4. We use the theory of super Jack polynomials ([SV05]) to pin down k_μ by comparing leading terms.



Main Results

Sketch of proof of Theorem B.

$$\begin{array}{ccc}
 \mathfrak{z} & \xrightarrow{\pi} & \mathfrak{D} \\
 \gamma \downarrow \wr & & \downarrow \wr \gamma^0 \\
 \Lambda(\mathfrak{h}^*) & \xrightarrow{\text{Res}} & \Lambda^0(\mathfrak{a}^*)
 \end{array}$$

1. Choose explicit coordinates on $\mathfrak{h}^*$ and $\mathfrak{a}^*$, and explicit generators of the algebra of supersymmetric polynomials ([Ste85]) to show the surjectivity of Res.
2. Diagram chase. The set of the highest $\mathfrak{h}$ -weights that guarantee to give a spherical irreducible $\mathfrak{g}$ -module is *Zariski dense* (we gave an independent argument. C.f. [AS15], a partial generalization of the Cartan–Helgason Theorem).

□

Main Results

Theorem C

The central element Z_μ acts on I_λ by 0 when $\lambda \not\geq \mu$.

Main Results

Theorem C

The central element Z_μ acts on I_λ by 0 when $\lambda \not\geq \mu$.

Sketch of the proof of Theorem C.

1. $I_\lambda = \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda \cong \mathfrak{S}(\mathfrak{p}^-) \otimes W_\lambda \cong \bigoplus (W_\mu^* \otimes W_\lambda)$ (as $\mathfrak{k}$ -modules).

Main Results

Theorem C

The central element Z_μ acts on I_λ by 0 when $\lambda \not\geq \mu$.

Sketch of the proof of Theorem C.

1. $I_\lambda = \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda \cong \mathfrak{S}(\mathfrak{p}^-) \otimes W_\lambda \cong \bigoplus (W_\mu^* \otimes W_\lambda)$ (as $\mathfrak{k}$ -modules).
2. Spherical: $I_\lambda^{\mathfrak{k}} \subseteq W_\lambda^* \otimes W_\lambda$ with $\dim I_\lambda^{\mathfrak{k}} = 1$.

Main Results

Theorem C

The central element Z_μ acts on I_λ by 0 when $\lambda \not\geq \mu$.

Sketch of the proof of Theorem C.

1. $I_\lambda = \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda \cong \mathfrak{S}(\mathfrak{p}^-) \otimes W_\lambda \cong \bigoplus (W_\mu^* \otimes W_\lambda)$ (as $\mathfrak{k}$ -modules).
2. Spherical: $I_\lambda^\mathfrak{k} \subseteq W_\lambda^* \otimes W_\lambda$ with $\dim I_\lambda^\mathfrak{k} = 1$.
3. Rep map $W_\mu \otimes I_\lambda^\mathfrak{k} \rightarrow I_\lambda$ has image homomorphic to W_μ .

Main Results

Theorem C

The central element Z_μ acts on I_λ by 0 when $\lambda \not\geq \mu$.

Sketch of the proof of Theorem C.

1. $I_\lambda = \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda \cong \mathfrak{S}(\mathfrak{p}^-) \otimes W_\lambda \cong \bigoplus (W_\mu^* \otimes W_\lambda)$ (as $\mathfrak{k}$ -modules).
2. Spherical: $I_\lambda^\mathfrak{k} \subseteq W_\lambda^* \otimes W_\lambda$ with $\dim I_\lambda^\mathfrak{k} = 1$.
3. Rep map $W_\mu \otimes I_\lambda^\mathfrak{k} \rightarrow I_\lambda$ has image homomorphic to W_μ .
4. $\text{Hom}_\mathfrak{k}(W_\mu, I_\lambda) = \{0\}$ for $\lambda \not\geq \mu$.

Main Results

Theorem C

The central element Z_μ acts on I_λ by 0 when $\lambda \not\geq \mu$.

Sketch of the proof of Theorem C.

1. $I_\lambda = \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda \cong \mathfrak{S}(\mathfrak{p}^-) \otimes W_\lambda \cong \bigoplus (W_\mu^* \otimes W_\lambda)$ (as $\mathfrak{k}$ -modules).
2. Spherical: $I_\lambda^\mathfrak{k} \subseteq W_\lambda^* \otimes W_\lambda$ with $\dim I_\lambda^\mathfrak{k} = 1$.
3. Rep map $W_\mu \otimes I_\lambda^\mathfrak{k} \rightarrow I_\lambda$ has image homomorphic to W_μ .
4. $\text{Hom}_\mathfrak{k}(W_\mu, I_\lambda) = \{0\}$ for $\lambda \not\geq \mu$.
5. $D_\mu = \sum \xi_i \eta_i$ for $\xi_i \in W_\mu^*$ and $\eta_i \in W_\mu$. So $D_\mu \cdot I_\lambda^\mathfrak{k} = \{0\} = \mathscr{D}_\mu \cdot I_\lambda^\mathfrak{k}$.

Main Results

Theorem C

The central element Z_μ acts on I_λ by 0 when $\lambda \not\geq \mu$.

Sketch of the proof of Theorem C.

1. $I_\lambda = \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda \cong \mathfrak{S}(\mathfrak{p}^-) \otimes W_\lambda \cong \bigoplus (W_\mu^* \otimes W_\lambda)$ (as $\mathfrak{k}$ -modules).
2. Spherical: $I_\lambda^\mathfrak{k} \subseteq W_\lambda^* \otimes W_\lambda$ with $\dim I_\lambda^\mathfrak{k} = 1$.
3. Rep map $W_\mu \otimes I_\lambda^\mathfrak{k} \rightarrow I_\lambda$ has image homomorphic to W_μ .
4. $\text{Hom}_\mathfrak{k}(W_\mu, I_\lambda) = \{0\}$ for $\lambda \not\geq \mu$.
5. $D_\mu = \sum \xi_i \eta_i$ for $\xi_i \in W_\mu^*$ and $\eta_i \in W_\mu$. So $D_\mu \cdot I_\lambda^\mathfrak{k} = \{0\} = \mathscr{D}_\mu \cdot I_\lambda^\mathfrak{k}$.
6. Z_μ also acts by 0. But $Z_\mu \in \mathfrak{Z}$ so it acts by 0 everywhere!

Main Results

Theorem C

The central element Z_μ acts on I_λ by 0 when $\lambda \not\geq \mu$.

Sketch of the proof of Theorem C.

1. $I_\lambda = \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda \cong \mathfrak{S}(\mathfrak{p}^-) \otimes W_\lambda \cong \bigoplus (W_\mu^* \otimes W_\lambda)$ (as $\mathfrak{k}$ -modules).
2. Spherical: $I_\lambda^\mathfrak{k} \subseteq W_\lambda^* \otimes W_\lambda$ with $\dim I_\lambda^\mathfrak{k} = 1$.
3. Rep map $W_\mu \otimes I_\lambda^\mathfrak{k} \rightarrow I_\lambda$ has image homomorphic to W_μ .
4. $\text{Hom}_\mathfrak{k}(W_\mu, I_\lambda) = \{0\}$ for $\lambda \not\geq \mu$.
5. $D_\mu = \sum \xi_i \eta_i$ for $\xi_i \in W_\mu^*$ and $\eta_i \in W_\mu$. So $D_\mu \cdot I_\lambda^\mathfrak{k} = \{0\} = \mathscr{D}_\mu \cdot I_\lambda^\mathfrak{k}$.
6. Z_μ also acts by 0. But $Z_\mu \in \mathfrak{Z}$ so it acts by 0 everywhere! □

The main thing is that I_λ has $\mathfrak{t}$ -highest weight and is infinite dimensional. We don't know by what “polynomial” $\mathscr{D}_\mu$ acts on $I_\lambda^\mathfrak{k}$ directly!