Supersymmetric Shimura operators and interpolation polynomials

Joint work with Siddhartha Sahi

Songhao Zhu

Georgia Institute of Technology
zhu.math@gatech.edu

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Structure

- Background
- Lie superalgebras
- Main Results
- Future Directions

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Symmetric functions naturally arise in representation theory.

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3. *CMS operators, Dunkl & Cherednik operators (related to DAHA)*
4. *Jack, Hall & Macdonald polynomials*
5. many more...

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We solved the Type A super analog:

Shimura Operators on Hermitian sym. sp. —superization→ Supersymmetric Shimura Operators of Hermitian sym. superpairs

Shimura Operators —[SZ19]→ Okounkov Polynomials

Supersymmetric Shimura Operators —[Zhu22, SZ23]→ Sergeev–Veselov Polynomials

Okounkov Polynomials —superization→ Sergeev–Veselov Polynomials

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2. Introduced certain G-invariant differential operators on a Hermitian $X := G/K$
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6. These are the even & symmetric polynomials with prescribed zeros (thus the word *interpolation*).
7. The theory of symmetric functions gives answers to Shimura's problem.

In context (frakturs)

Let $X := G/K$ be as above. Change the point of view to (complexified) Lie algebras.

1. $(\mathfrak{g}, \mathfrak{k})$: Harish-Chandra decomposition [$(-1, 0, 1)$-grading]

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2. $\mathfrak{D} = \mathfrak{D}(X)$: space of invariant differential operators on X.
$\mathfrak{U} := \mathfrak{U}(\mathfrak{g})$; $\quad \mathfrak{U}^\mathfrak{k} := \mathfrak{Z}_\mathfrak{k}(\mathfrak{U})$; $\quad (\mathfrak{U}\mathfrak{k})^\mathfrak{k} := \mathfrak{U}\mathfrak{k} \cap \mathfrak{U}^\mathfrak{k}$.
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4. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$: Iwasawa decomposition (Group: $G = KAN$). Here $\mathfrak{a}$ is maximally abelian in $\mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{p}^+$. May identify $\mathfrak{S}(\mathfrak{a})$ with $\mathfrak{P}(\mathfrak{a}^*)$.
5. $\gamma^0 : \mathfrak{D} \to \Lambda \subseteq \mathfrak{P}(\mathfrak{a}^*)$: the Harish-Chandra isomorphism. Good map! Essentially a symmetry-preserving projection, shifted by the Weyl vector ρ

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Schmid decomposition ([Sch70, FK90]) for $\mathfrak{k}$-mods (multiplicity-free):

$$\mathfrak{S}^d(\mathfrak{p}^+) = \bigoplus_{\lambda \in \mathscr{P}^d(n)} W_\lambda, \ \mathfrak{S}^d(\mathfrak{p}^-) = \bigoplus_{\lambda \mathscr{P}^d(n)} W_\lambda^*.$$

Here we choose a form to identify $\mathfrak{p}^-$ *with* $(\mathfrak{p}^+)^*$*. The highest weights are parametrized using the HC strongly orthogonal roots.*

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Shimura Operators

$$\begin{array}{c} \mathrm{End}_{\mathfrak{k}}(W_\lambda) \cong (W_\lambda^* \otimes W_\lambda)^{\mathfrak{k}} \hookrightarrow \left(\mathfrak{S}\left(\mathfrak{p}^-\right) \otimes \mathfrak{S}\left(\mathfrak{p}^+\right)\right)^{\mathfrak{k}} \to \mathfrak{U}^{\mathfrak{k}} \to \mathfrak{D} \\ 1 \longmapsto D_\lambda \mapsto \mathscr{D}_\lambda \end{array}$$

Call $\mathscr{D}_\lambda$ the *Shimura operator associated with λ*.

A **basis** parameterized by **partitions**! Nice images of 1's.

Okounkov Polynomials

$\Lambda := \mathbb{C}[x_1, \ldots, x_n]^{S_n \ltimes \mathbb{Z}_2^n}$ (ring of even symmetric polynomials)
$\rho := (\rho_1, \ldots, \rho_n)$, $\rho_i := \tau(n-i) + \alpha$. τ, α: parameters.

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Theorem-Definition [Oko98, OO06], c.f. [SZ19]

The Okounkov polynomial $P_\mu(x_1, \dots, x_n; \tau, \alpha)$ is the unique polynomial in Λ satisfying

1. $\deg P_\mu = 2|\mu|$;
2. $P_\mu(\lambda + \rho) = 0$ for $\lambda \not\supseteq \mu$ [**the vanishing properties**];
3. Some normalization condition (at $\mu + \rho$).

1. They are interpolation polynomials (interpolated by zeros);
2. ρ can be specialized to the half sum of positive roots for a restricted root system of Type BC [The case for Hermitian X];
3. For the "usual" Type A symmetry, there are Knop–Sahi polynomials [KS96].

An Example

Consider the one-row partition (l) for $l \in \mathbb{N}$ (so $(k) \not\supseteq (l)$ means $k < l$).
$n = 1$ $\rho = \alpha$. We have [SZ19]:

$$P_{(l)}(x;\tau,\alpha) = \prod_{i=0}^{l-1}(x^2 - (i+\alpha)^2) = \prod_{i=0}^{l-1}(x+\alpha-i)(x-\alpha+i).$$

- Trivially symmetric;
- Degree $2l$;
- Vanishes at $(k+\alpha)$ for $k < l$.

For (l^n), we have

$$P_{(l^n)}(x;\tau,\alpha) = \prod_{i=0}^{l-1}\prod_{j=1}^{n}(x_j^2 - (i+\alpha)^2)$$

Classical result

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Let V_μ be the irreducible $\mathfrak{g}$-module (of the same highest weight as W_μ). Then by the Cartan–Helgason Theorem, V_μ has a spherical vector $v^{\mathfrak{k}}$, i.e. $\mathfrak{k}.v^{\mathfrak{k}} = 0$. $D_\lambda \in \mathfrak{U}^{\mathfrak{k}}$ ($\mathscr{D}_\lambda \in \mathfrak{D}$) acts on $v^{\mathfrak{k}}$ as $\gamma^0(\mathscr{D}_\lambda)(\mu + \rho)$, hence the word spectrum/eigenvalue!

Lie Superalgebras

Definition

A *Lie superalgebra* is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear map $[-,-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying

1. $[X, Y] = -(-1)^{|X||Y|}[Y, X]$
2. $[[X, Y], Z] = [X, [Y, Z]] - (-1)^{|X||Y|}[Y, [X, Z]]$

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$$\mathfrak{gl}_{\overline{0}}\colon \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) \qquad \mathfrak{gl}_{\overline{1}}\colon \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right) \qquad [X, Y] := XY - (-1)^{|X||Y|}YX.$$

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Bad news for LSA

No Weyl's theorem on complete reducibility; Borels are not conjugates; *isotropic* (restricted) roots...

Supersymmetric Shimura operators

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$\mathscr{H} = \mathscr{H}(p,q) := \{\lambda : \lambda_{p+1} \leq q\}$ (hook partitions). $\mathscr{H}^d := \{\lambda \in \mathscr{H} : |\lambda| = d\}$.
Super analog of the Schmid decomposition: ([CW01, SSS20])

$$\mathfrak{S}^d(\mathfrak{p}^+) = \bigoplus_{\lambda \in \mathscr{H}^d} W_\lambda, \; \mathfrak{S}^d(\mathfrak{p}^-) = \bigoplus_{\lambda \in \mathscr{H}^d} W_\lambda^*.$$

W_λ are of Type `M`, and $\dim \operatorname{End}_{\mathfrak{k}}(W_\lambda) = 1$. (Super version of Schur's Lemma may include the parity twist.) Set $\mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$.

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Proposition-Definition [SV09]

For each $\mu \in \mathscr{H}$, there is a unique polynomial $J_\mu \in \Lambda^0$ of degree $2|\mu|$ s.t.

$$J_\mu(\overline{\lambda} + \rho) = 0, \quad \text{for all } \lambda \not\supseteq \mu,\ \lambda \in \mathscr{H}$$

and that $J_\mu(\overline{\mu} + \rho)$ is certain explicit non-zero constant.

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2. $\mu = (2)$, $\lambda = (1^n)$, and $\overline{\lambda} + \rho = (1, 2n - 1)$, $J_{(2)} \propto (x^2 - y^2)(x^2 - 1)$.

Main Results

Theorem A (Sahi–Z. [SZ23])

We have $\gamma^0(\mathscr{D}_\mu) = k_\mu J_\mu$ *where* $k_\mu = (-1)^{|\mu|} \prod_{(i,j)\in\mu} \left(\mu_i - j + \mu'_j - i + 1\right)$.

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The main thing is to show the vanishing properties. Need two other results.
Let the center of $\mathfrak{U}$ be $\mathfrak{Z}$. Then $\mathfrak{Z} \subseteq \mathfrak{U}^{\mathfrak{k}}$ and we have $\pi : \mathfrak{Z} \hookrightarrow \mathfrak{U}^{\mathfrak{k}} \twoheadrightarrow \mathfrak{D}$. Establish the commutative diagram:

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{\pi} & \mathfrak{D} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma^0} \\ \mathfrak{P}(\mathfrak{h}^*) & \xrightarrow{\texttt{Res}} & \mathfrak{P}(\mathfrak{a}^*) \end{array}$$

Main Results

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{\pi} & \mathfrak{D} \\ \gamma\downarrow & & \downarrow\gamma^0 \\ \mathfrak{P}(\mathfrak{h}^*) & \xrightarrow{\mathsf{Res}} & \mathfrak{P}(\mathfrak{a}^*) \end{array} \qquad \begin{array}{ccc} Z_\mu & \overset{\pi}{\longmapsto} & \mathscr{D}_\mu \\ \gamma\downarrow & & \downarrow\gamma^0 \\ \gamma(Z_\mu) & \overset{\mathsf{Res}}{\longmapsto} & \gamma^0(\mathscr{D}_\mu) \end{array}$$

Theorem B (Sahi–Z. [SZ23])

The map π is surjective. In particular, there exists $Z_\mu \in \mathfrak{Z}$ such that $\pi(Z_\mu) = \mathscr{D}_\mu$. (So $\mathscr{D}_\mu = \pi(D_\mu)$ can be captured by some central element!)

Let $I_\lambda := \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{q})} W_\lambda$ be the generalized Verma module for $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$.

Theorem C (Sahi–Z. [SZ23])

The central element Z_μ acts on I_λ by 0 when $\lambda \not\supseteq \mu$.

Future Directions

- Other pairs (Jordan superalgebras+TKK construction) c.f. [SSS20].

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Future Directions

- Other pairs (Jordan superalgebras+TKK construction) c.f. [SSS20].
- No full generalization of Cartan–Helgason theorem in the super scenario. The only partial result is obtained by Alldridge and Schmittner [AS15]. Not enough V_λ are guaranteed to be spherical. This is used in the old method [Zhu22]. *Some progress!*
- Differential operators (on (possibly Laurent) polynomials) $\sim$ symmetric polynomials. This Shimura theory should in principle help us to explicitly write down a family of commuting differential operators (also commuting with the CMS operators) *Ongoing! Nothing wrong so far...*

Scope: Shimura and friends

	Shimura	Capelli	quadratic Capelli
☺	[SZ19]	[KS93, Sah94]	[SS19]
$\mathbb{Z}_2$	[Zhu22, SZ23]	[SSS20]	?
q	?	[LSS22]	?
$\mathbb{Z}_2, q$	?	?	?

Thank you!

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Set up

Fix $\mathfrak{g} = \mathfrak{gl}(2p|2q)$ and $\mathfrak{k} = \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q)$. Embed $\mathfrak{k}$ into $\mathfrak{g}$:

$$\left(\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right), \left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right) \right) \mapsto \left(\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ \hline C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{array} \right)$$

Here $\mathfrak{p}^{+}$ (resp. $\mathfrak{p}^{-}$) consists of matrices with non-zero entries only in the upper right (resp. bottom left) sub-blocks in each of the four blocks.

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Here $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) consists of matrices with non-zero entries only in the upper right (resp. bottom left) sub-blocks in each of the four blocks.
Technical details:

1. Let $J := \frac{1}{2}\,\mathrm{diag}(I, -I, I, -I)$, and $\theta := \mathrm{Ad}\exp(i\pi J)$. Then θ has fixed point subalgebra $\mathfrak{k}$.
2. The standard diagonal Cartan is denoted as $\mathfrak{t}$ (in both $\mathfrak{g}$ and $\mathfrak{k}$, "max. compact")
3. Fix a θ-stable, maximally split Cartan $\mathfrak{h}$ containing $\mathfrak{a}$, a maximal toral subalgebra in $\mathfrak{p}_{\overline{0}}$ (for Iwasawa decomp.).

Supersymmetric polynomials

For polynomials, supersymmetry = symmetry + translational invariance.

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For polynomials, supersymmetry = symmetry + translational invariance.
More precisely, in the Type BC situation:

Definition

Let $\{\epsilon_i\}_{i=1}^p \cup \{\delta_j\}_{j=1}^q$ be the standard basis for $V = \mathbb{C}^{p+q}$. Denote the coordinate functions for this basis as x_i and y_j. A polynomial $f \in \mathfrak{P}(V) = \mathbb{C}[x_i, y_j]$ is said to be *even supersymmetric* if :

(i) f is symmetric in x_i and y_j separately and invariant under sign changes of x_i and y_j;

(ii) $f(X + \epsilon_i - \delta_j) = f(X)$ if $x_i + y_j = 0$ for $i = 1, \ldots, p$, $j = 1, \ldots, q$.

We denote the subring of even supersymmetric polynomials as $\Lambda^0(V)$.

!! (i) $\iff$ invariance under $(S_p \ltimes \mathbb{Z}_2^n) \times (S_q \ltimes \mathbb{Z}_2^n)$ but (ii) is no longer group invariance (not even linear).

In fact, one may use a suitable Weyl *groupoid* action to capture (i & ii).